An Affinity for Affine Quantum Gravity

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Dictionary Definitions

- Affinity

  1. a natural liking for or attraction to a person, thing, idea, etc.

- Affine

  3. of or pertaining to a transformation that maps parallel lines to parallel lines and finite points to finite points.

Affine Quantum Gravity

- Kinematical variables
  - Metric positivity
- Initial representation
  - Without constraints
- Quantum constraints
  - Projection operator method
- Imposition of constraints
  - Functional integral representation

Kinematical variables

- Affine commutation relations

$$g_{ab}(x),\ \underline{\{g_{ab}(x)\}>0}\ , \qquad \pi^a_b(x)=\pi^{ac}(x)g_{cb}(x)$$

$$[\hat{\pi}^a_b(x),\hat{\pi}^c_d(y)]=(i/2)[\delta^a_d\hat{\pi}^c_b(x)-\delta^c_b\hat{\pi}^a_d(x)]\delta(x,y)$$

$$[\hat{g}_{ab}(x),\hat{\pi}^c_d(y)]=(i/2)[\delta^c_a\hat{g}_{bd}(x)+\delta^c_b\hat{g}_{ad}(x)]\delta(x,y)$$

$$[\hat{g}_{ab}(x),\hat{g}_{cd}(y)]=0$$

$$U(\gamma)=\exp[-i\int\gamma^b_a\hat{\pi}^a_b d^3y]\ , \qquad M^c_a(x)=\{\exp[\gamma(x)/2]\}^c_a$$

$$U(\gamma)^{-1}\hat{g}_{ab}(x)U(\gamma)=M^c_a(x)\hat{g}_{cd}(x)M^d_b(x)$$

Initial representation

- Affine coherent states

$$|\pi, g\rangle = e^{i\int \pi^{cd}\hat{g}_{cd}d^3y} e^{-i\int \gamma_a^b \hat{\pi}_b^a d^3y} |\eta\rangle$$

$$\langle\eta|\hat{g}_{cd}(x)|\eta\rangle = \tilde{g}_{cd}(x) \quad , \qquad \langle\eta|\hat{\pi}_b^a(x)|\eta\rangle = 0$$

$$\langle\pi'', g''|\pi', g'\rangle = \exp\Big\{-2\int b(x)d^3x \times \ln\Big(\frac{\det\{(g''^{ab}+g'^{ab})/2+ib(x)^{-1}(\pi''^{ab}-\pi'^{ab})/2\}}{[\det\{g''^{ab}\}\det\{g'^{ab}\}]^{1/2}}\Big)\Big\}$$

$$g_{ab}(x) = M_a^c(x)\tilde{g}_{cd}(x)M_b^d(x) \quad , \qquad M_a^c(x) = \{\exp[\gamma(x)/2]\}_a^c$$

Quantum constraints

- Enforcing the constraints

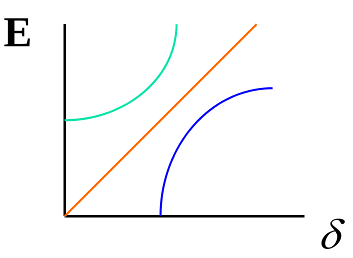
$$\varphi_\alpha(p,q)=0 \quad , \qquad\qquad \sum_{\alpha=1}^{A} \varphi_\alpha(p,q)^2 = 0$$

- Dirac: $\Phi_\alpha(P,Q)\,|\psi_{\text{phys}}\rangle = 0$, (limited use)
- POM: $\mathbf{E}=\mathbf{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta(\hbar)^2)$, $\mathbf{H}_{\text{phys}} = \mathbf{E}\mathbf{H}$

(1) $\mathbf{E}(J_1^2 + J_2^2 + J_3^2 \le \hbar^2/2)$

(2) $\mathbf{E}(P^2 + Q^2 \le \hbar)$

(3) $\mathbf{E}(Q^2 \le \delta^2) = \mathbf{E}(-\delta < Q < \delta)$



$$\lim_{\delta\to 0} \langle p'', q''|\mathbf{E}|p', q'\rangle / \langle \eta|\mathbf{E}|\eta\rangle = \langle\langle p'', q''|p', q'\rangle\rangle$$

Quantum constraints (2)

- Non-standard constraints

(4) $\mathbf{E}(Q^2+Q^2\le\delta^2)=\mathbf{E}(Q^2+Q^2<\delta^2)$

$\lim_{\delta\to 0}\langle p'',q''|\mathbf{E}|p',q'\rangle/\langle\eta|\mathbf{E}|\eta\rangle=\langle\langle p'',q''|p',q'\rangle\rangle$

(5) $\mathbf{E}(Q^{2\Omega}\le\delta^2)\quad,\qquad \Omega>1$

$\lim_{\delta\to 0}\langle p'',q''|\mathbf{E}|p',q'\rangle/\langle\eta|\mathbf{E}|\eta\rangle=\langle\langle p'',q''|p',q'\rangle\rangle$

(6) $\mathbf{E}(Q^2(Q-1)^4\le\delta^2)=\mathbf{E}_0(Q^2\le\delta^2)+\mathbf{E}_1((Q-1)^4\le\delta^2)$

$A(l'';l')=\langle l''|\mathbf{E}_0|l'\rangle/\sqrt{\langle\eta_0|\mathbf{E}_0|\eta_0\rangle}+\langle l''|\mathbf{E}_1|l'\rangle/\sqrt{\langle\eta_1|\mathbf{E}_1|\eta_1\rangle}$

$\lim_{\delta\to 0}\int A(l'';l)\langle l|\mathbf{E}|\tilde{l}\rangle A(\tilde{l};l')\;\;dl\,d\tilde{l}\;=\langle\langle l''|l'\rangle\rangle$

Imposition of Constraints

- Path integral

$$\mathbf{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta^2) = \int \mathbf{T} e^{-i\int \lambda^\alpha \Phi_\alpha dt} \, DR(\lambda)$$

universal!

$$\langle q'' | \mathbf{E} e^{-i(\mathbf{E}\hat{H}\mathbf{E})T} \mathbf{E} | q' \rangle = M \int e^{i\int [p\dot{q} - H(p,q) - \lambda^\alpha \varphi_\alpha(p,q)]dt} \, DpDqDR(\lambda)$$

$$\langle p'', q'' | \mathbf{E} e^{-i(\mathbf{E}\hat{H}\mathbf{E})T} \mathbf{E} | p', q' \rangle = \lim_{\nu \to \infty} \widetilde{M}_\nu \int e^{i\int [p\dot{q} - H(p,q) - \lambda^\alpha \varphi_\alpha(p,q)]dt}$$
$$\times \exp\{-(1/2\nu)\int [\dot{p}^2 + \dot{q}^2] dt\} \, DpDqDR(\lambda)$$

Initial representation

- Affine coherent states

$$|\pi, g\rangle = e^{i\int \pi^{cd}\hat{g}_{cd}d^3y} e^{-i\int \gamma_a^b \hat{\pi}_b^a d^3y} |\eta\rangle$$

$$\langle\eta|\hat{g}_{cd}(x)|\eta\rangle = \tilde{g}_{cd}(x) \quad , \qquad \langle\eta|\hat{\pi}_b^a(x)|\eta\rangle = 0$$

$$\langle\pi'', g''|\pi', g'\rangle = \exp\Big\{-2\int b(x)d^3x \times \ln\Big(\frac{\det\{(g''^{ab}+g'^{ab})/2+ib(x)^{-1}(\pi''^{ab}-\pi'^{ab})/2\}}{[\det\{g''^{ab}\}\det\{g'^{ab}\}]^{1/2}}\Big)\Big\}$$

$$g_{ab}(x) = M_a^c(x)\tilde{g}_{cd}(x)M_b^d(x) \quad , \qquad M_a^c(x) = \{\exp[\gamma(x)/2]\}_a^c$$

Imposition of Constraints (2)

- Functional integral

The Heart of the Matter

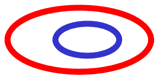
$$\langle \pi'', g'' | \mathbf{E} | \pi', g' \rangle = \lim_{\nu \to \infty} M_\nu \int \exp\{-i\int [g_{ab}\dot{\pi}^{ab} + N^a H_a + NH]dtd^3x\}$$
$$\times \exp\{-(1/2\nu)\int [b^{-1}(x) g_{ab} g_{cd} \dot{\pi}^{bc}\dot{\pi}^{da} + b(x) g^{ab} g^{cd} \dot{g}_{bc}\dot{g}_{da}]dtd^3x\}$$
$$\times [\prod_{t,x} \prod_{a \geq b} d\pi^{ab}(t,x) dg_{ab}(t,x)] \;\; DR(N^a, N)$$

Further Issues

- Nonrenormalizability?
  - Hard-core interpretation
- Appearance of Time?
  - Soluble examples
- Removal of Cutoffs?
  - Choose proper representation
  - Reduce δ as necessary

Hard-core interpretation

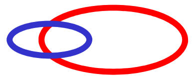
Classical action: $I = \int (\underline{\dot{x}^2/2} - \underline{\lambda x^4})dt$



$$\langle x'', T | x', 0 \rangle = \lim_{\lambda \to 0} \int \exp[-\int (\dot{x}^2/2 + \lambda x^4)dt]Dx$$

$$= \frac{1}{\sqrt{2\pi T}} \exp[-(x'' - x')^2/2T]$$

Classical action: $I = \int (\underline{\dot{x}^2/2} - \lambda \underline{/x^4})dt$

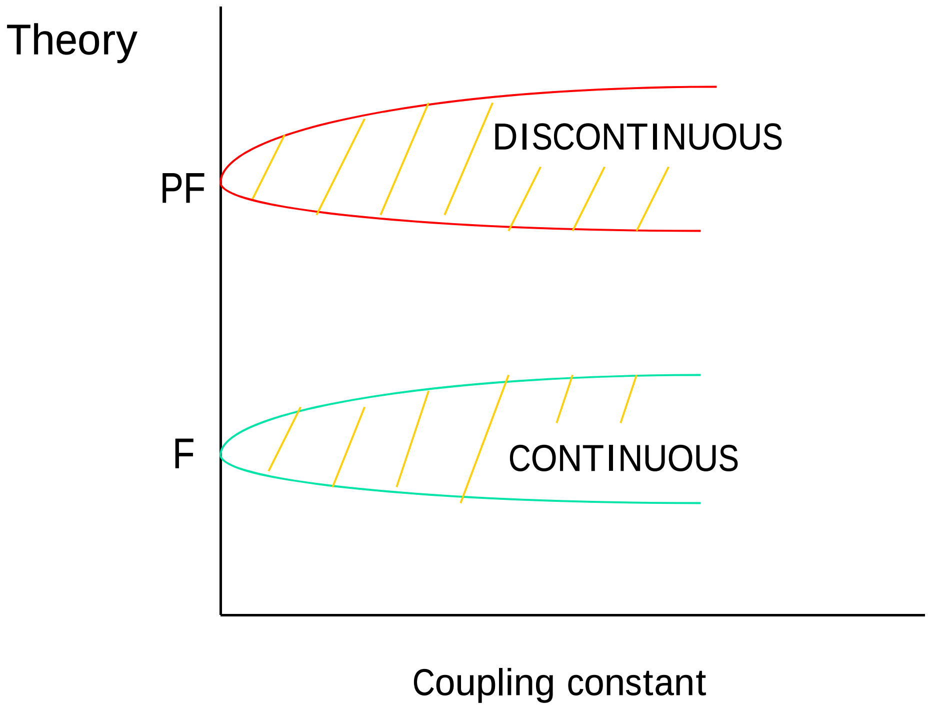


$$\langle x'', T | x', 0 \rangle = \lim_{\lambda \to 0} \int \exp[-\int (\dot{x}^2/2 + \lambda/x^4)dt]Dx$$

$$= \frac{\theta(x''x')}{\sqrt{2\pi T}} \left\{ \exp[-(x'' - x')^2/2T] - \exp[-(x'' + x')^2/2T] \right\}$$

Hard-core interpretation (2)

- Free and pseudo-free theories



Hard-core interpretation (3)

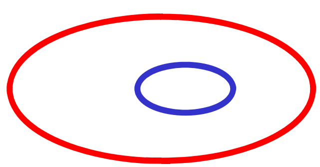
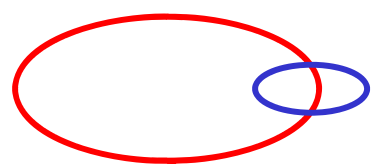
- Scalar field theory

$$\lim_{\lambda\to 0} N_\lambda \int \exp\left(\int \{ h\varphi - \underline{[(\nabla\varphi)^2 + m^2\varphi^2]}/2 - \underline{\lambda\varphi^4} \} \underline{d^3x} \right) D\varphi$$

$$= N_0 \int \exp\left(\int \{ h\varphi - [(\nabla\varphi)^2 + m^2\varphi^2]/2 \} d^3x \right) D\varphi \equiv \underline{S_0(h)}$$

$$\lim_{\lambda\to 0} N_\lambda \int \exp\left(\int \{ h\varphi - \underline{[(\nabla\varphi)^2 + m^2\varphi^2]}/2 - \underline{\lambda\varphi^4} \} \underline{d^5x} \right) D\varphi$$

$$= N_0' \int \underline{X(\varphi)} \exp\left(\int \{ h\varphi - [(\nabla\varphi)^2 + m^2\varphi^2]/2 \} d^5x \right) D\varphi \equiv \underline{S_0'(h)}$$

3  5 

Hard-core interpretation (4)

- Scalar field theory

$$S_\lambda(h) = N_\lambda \int \exp\left(\int \{h\varphi - [(\nabla\varphi)^2 + m^2\varphi^2]/2 - \lambda\varphi^4\} d^n x \right) D\varphi$$

$$S_0(h) = N_0 \int \exp\left(\int \{h\varphi - [(\nabla\varphi)^2 + m^2\varphi^2]/2\} d^n x \right) D\varphi$$

When $h = 0$, $S_\lambda(0) = S_0(0) = 1$

Does $\lim_{\lambda\to 0} S_\lambda(h) = S_0(h)$ for all h?

$$\{\int \varphi^4 d^n x\}^{1/2} / \int [(\nabla\varphi)^2 + m^2\varphi^2] d^n x \le C$$

If $n \le 4$, $C = 4/3$ $(Yes - \text{Renormalizable})$

If $n \ge 5$, $C = \infty$ $(No - \text{Nonrenormalizable})$

Summary: AQG

- *Preserve metric positivity*
  Affine kinematical variables
- *Gravitational anomaly*
  Projection operator method
- *Functional integral formalism*
  Continuous-time regularization
- *Nonrenormalizable theory*
  Hard-core interaction

Thank you !