Cohomologies in Superstring Field Theories and D-branes Properties

I.Ya. Aref'eva

Steklov Mathematical Institute, Moscow

"Gauge Fields. Yesterday, Today,Tomorrow",

conference dedicated to 70th birthday of A.A.Slavnov,

Moscow, 20-23 January, 2010

Q and Slavnov-Taylor identities

$$[T^a, T^b] = t^{abc} T^b, \quad Q = c^a T^a - \frac{1}{2} t^{abc} c^a \bar{c}^b c^c, \quad \{c^a, \bar{c}^b\} = \delta^{a,b}$$

Coboundary operator

$$d\omega^n(x_1, \ldots x_{n+1}) = \sum_{i=1}^{n+1} (-)^{i-1} d^{(0)}(x_i)\, \omega^n(x_1, \ldots \hat{x}_i \ldots x_{n+1})$$

$$+ \sum_{i<j}^{n+1} (-)^i\, \omega^n(x_1, \ldots \hat{x}_i \ldots [x_i, x_j] \ldots\ x_{n+1})$$

$\omega^n \in C^n(\mathcal{G}, \mathcal{A}) - n -$ linear forms

$\mathcal{G}$ an algebra, $\mathcal{A}$ a module

$d^{(0)} : \mathcal{G} \to End(\mathcal{A})$

C.Chevalley, S.Eilenberg, "Cohomologt Theory of Lie Groups and Lie Algebras" Transaction AMS, 63(1948) 85

BRST –operator = Chevalley-Eilenberg coboundary operator

$$Q_{BRST} = d$$

Slavnov-Taylor identities and Q

$$L_{eff} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 + \overline{c}^a M^{ab} c^b$$

Faddeev-Popov ghosts

$$A^a_\mu(x) \rightarrow A^a_\mu(x) + [\nabla_\mu c(x)]^a \varepsilon$$

$$c^a(x) \rightarrow c^a(x) - \frac{1}{2} t^{abd} c^b(x)\, c^d(x)\, \varepsilon$$

$$\overline{c}^a(x) \rightarrow \overline{c}^a(x) + \frac{1}{2} [\partial_\mu A^a_\mu(x)] \varepsilon$$

$$\varepsilon^2 = 0, \quad \varepsilon c + c \varepsilon = 0 \quad \varepsilon \overline{c} + \overline{c} \varepsilon = 0$$

A.A.Slavnov,L.D.Faddeev, Introduction to quantum theory gauge theories, Nauka, 1978

From the letter of R.Stora:

It is Andrej's short paper which triggered what is now commonly baptized as "BRST", to the extent that we called it the Slavnov symmetries, because, had he performed the calculation of his paper in general, he would have bumped into it!

R. Stora

Q and Slavnov-Taylor identities

$$[T^a, T^b] = t^{abc} T^b$$

$$Q = c^a T^a - \frac{1}{2} t^{abc} c^a \overline{c}^b c^c,$$

$$\{c^a, \overline{c}^b\} = \delta^{a,b}$$

Local gauge theories: T^a

generators of gauge transformations

Strings: $T^a \Rightarrow V_n$

generators of Virasoro algebra

Outlook

- **Short introduction to SFT**
- **Cohomologies of Q around perturbative vacuum in bosonic SFT**
- **Cohomologies of Q around nonperturbative vacuum in bosonic SFT and D-brane decay** (homotopy operator)
- **Cohomologies of Q in SuperSFT**
- **Chern–Simons vs. WZWN in SSFT**

Why string field theory ?

In the first quantized approach of string theory we have

- rules how to compute on-shell scattering amplitudes.
- a problem with an explicit covering of supermoduli space
- no space for off-shell phenomena (solitons, Higgs mechanism etc.)

What is string field theory (SFT)?

- A field theory for all the modes of a string
- SFT is the most straightforward generalization of what is done in particle physics

  (gauge invariance, Higgs phenomena)

What can SFT be useful for ?

- Systematic computations of scattering amplitudes
- To find new backgrounds in string theory
- Study processes involving branes

Main ingredients in SFT (bosonic)

A string field Ψ

- It can be understood either as a classical functional of the open string configurations

$$\Psi[x(\sigma), c(\sigma), b(\sigma)]$$

or

- *as a vector in the Fock space of states of the open string theory*

$$|\Psi\rangle = \phi(x)c_1|0\rangle + A_\mu(x)\alpha_{-1}^{\mu}c_1|0\rangle + iB(x)c_0|0\rangle + \cdots$$

Main ingredients in SFT (bosonic)

BRST charge *Q*

The BRST charge *Q is the same as for the first quantized string theory*

$$Q_{\mathrm{B}} = \oint \frac{dz}{2\pi i}\left(cT^{\mathrm{m}} + bc\partial c + \frac{3}{2}\partial^2 c\right) \qquad Q\,\Psi = 0$$

The spectrum of physical states around the perturbative vacuum is given by the BRST cohomology

$$\mathrm{Ker}\ \mathrm{Q}_1 / \mathrm{Im}\ \mathrm{Q}_0$$

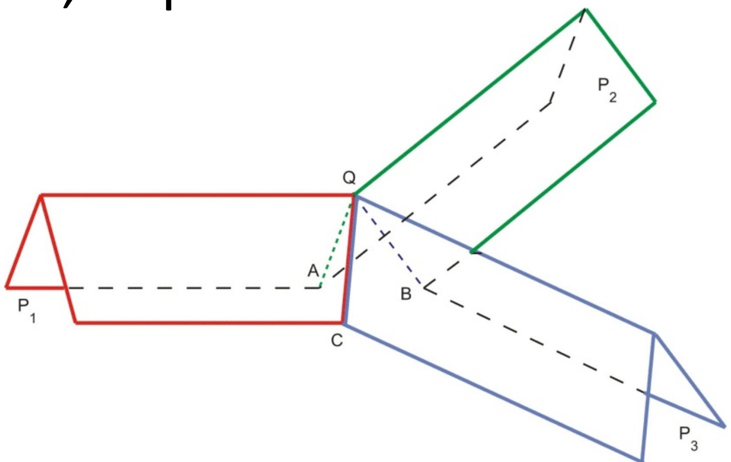
Q_n describes the action of the BRST operator Q on a string field of ghost number n

Main ingredients in SFT

Star Product

- Physically it represents the string interaction, that is the process of two strings coming together to form a third string.
- More precisely the product of two string fields represents the process of identifying the right half of the first string with the left half of the second string and integrating over the overlapping degrees of freedom, to produce a third string which corresponds to

$$|\Phi_1 > * |\Phi_2 > = |\Phi_1 * \Phi_2 >$$



Main ingredients in SFT(bosonic)

Action

- (Witten 1986)

$$S = \frac{1}{2} < \Phi, Q\,\Phi > + \frac{1}{3} < \Phi, \Phi * \Phi >$$

$$< \Phi | = BPZ\,|\Phi >$$

This action is clearly reminiscent of the Chern–Simons action in 3D.

Chern-Simons

- M is 3-manifold, principle G-bundle over M and A is one -form with values in G,
- action

$$S = \frac{k}{4\pi} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

k-non-integer,
Witten, 1001.2933

E.O.M.

$$F = 0, \quad F = dA + A \wedge A$$

Flat connections

Theorem. Flat connections of principal G-bundles over M are entirely determined by holonomies around noncontractive cycles on the base M

or

Flat connections of principal G-bundles over M are in one to one correspondence with equivalence classes of homomorphism from the fundamental group of M to G up to conjugation, i.e.

$$\{\pi(M) \to G\} \Leftrightarrow \{A : dA + A \wedge A = 0\}$$

Solution to E.O.M.

$$Q\Phi + \Phi * \Phi = 0$$

Two Lemmata

Lemma 2

If operator Q *has trivial cohomologies, i.e. all solutions to* $Q\phi = 0$ *have form* $\phi = Q\chi$*, then equation*

$$Q\Phi + \Phi * \Phi = 0$$

has only trivial solutions (pure gauge)

$$\Phi = (QU) * U^{-1},$$

where $U * U^{-1} = \mathcal{I}$.

$\mathcal{I}$ **is an unit Gross-Jevicki string operator**

Proof

Since Q has hero cohomology, according Lemma 1,

$$QA = \mathcal{I}$$

we have

$$Q(\mathcal{I} + A * \Phi) = (\mathcal{I} + A * \Phi) * \Phi \ .$$

Let $U = \mathcal{I} + A * \Phi$, then

$$QU = U\Phi \ ,$$

so that

$$\Phi = U^{-1}QU \ .$$

Two Lemmata

Lemma 1

The cohomology of a BRST operator Q vanishes if and only if there exists a string field A such that

$$QA = \mathcal{I}.$$

Here $\mathcal{I}$ identity operator in an operator algebra, such that

$$Q\mathcal{I} = 0$$

A is a homotopy operator

Proof.
First, suppose that Q has no cohomology.

$$Q\mathcal{I} = 0 \;\Rightarrow\; \mathcal{I} = QA$$

Now suppose, instead, that we have a state A such that

$$QA = \mathcal{I}$$

Suppose we also have some Q-closed state Λ such that

$$Q\Lambda = 0.$$

Then

$$Q(A * \Lambda) = (QA) * \Lambda + (-)^{|A|} A * \underbrace{Q\Lambda}_{=0} = \mathcal{I} * \Lambda = \Lambda,$$

so that Λ is Q-exact.

I.Numerical Solutions to SFT(2000-2003)

- Bosonic case. Record calculations by

  Moeller, Rastelli, Zwiebach

- Fermionic case,
- B(+,-) theory; Berkovits,Sen, Zwiebach (2000)

  ABKM = IA, Belov,Koshelev, Medvedev (2001)

II. 2005-2008 M.Schnabl and the following

- Formal pure gauge solutions

$$\Phi = Q\phi \frac{1}{1-\phi}$$

here we understand the fraction perturbatively and all product are $\star$-products.

Schnabl; Okawa; Erler; I.A., Gorbachev, Medvedev; Fuchs,Kroyter; Takahashi; IA, Grigoriev, Gormachev, Khromov, Malchev,Medvedev

A regularization required

III. 2009 –now

Technical remarks

- Star product is isomorphic to operator product

$$|\Phi_1 > * |\Phi_2 > = |\Phi_1 \, e^K \, \Phi_2 >$$

- Isomorphism

$$|\Phi > \to |\hat{\Phi} > = |e^{K/2}\Phi \;\; e^{K/2} >$$

$$|\hat{\Phi}_1 > * |\hat{\Phi}_2 > = |\widehat{\Phi_1 \Phi_2} >$$

A simplest solution to E.O.M.

$$Q\Phi + \Phi * \Phi = 0$$

$$[K, c] = \partial c$$

$$QK = 0,$$

$$Qc = cKc$$

$$\Phi = c - cK$$

$$Q\Phi \;\; = cKc - cKcK$$

$$\Phi * \Phi = -cKc + cKcK$$

This solution unfortunately does not obey regularity conditions.

A cured solution to E.O.M.

Erler, Schnabl, 2009

$$Q\Phi + \Phi * \Phi = 0$$

$$[K,B] = 0, \quad \{B,c\} = 1,$$

$$B^2 = 0, \quad c^2 = 0,$$

$$[K,c] = \partial c$$

$$QK = 0, \qquad QB = K$$

$$Qc = cKc$$

$$\Phi = cB(1+K)c\frac{1}{1+K}$$

A nice feature of the ES solution is that it does not have phantom terms and it is easy to compute the energy.

NS GSO(+)

$$\Phi_{RG} = (c + cKBc + B\gamma^2)\frac{1}{1+K}$$

Gorbachev, 2009

NS GSO(+)
GSO(-)

Work in progress

IA,Gorbachev, 2010

Kinetic operator around the new vacuum (physical spectrum in new vacuum)

$$Q_{\Phi} = Q + \{\Phi,\ \cdot\}$$

To see whether there is any cohomology or not, around the solution, we may attempt $Q+\{\Phi,\ \cdot\}$ to construct a homotopy operator, such that

$$\{Q_{\Phi}, A\} = 1$$

Such an operator does exist

$$A = B\frac{1}{1+K}$$

- **The main difference in the two BRST operators:**

  Q has nontrivial cohomologies

  Q_Φ **has NO nontrivial cohomologies**

- **No open string mode at the tachyon vacuum (3^{rd} Sen's conjecture)**

  Ellwood and Schnabl, 08

SuperSFT

- Picture

$$S = \frac{1}{2} << \Phi, Q\Phi >> + \frac{1}{3} << \Phi, \Phi \star \Phi >>$$

$$<< \dots >> = < Y_{-2} .. >$$

Φ belong "the small Hilbert space"

η_0, ξ_0 are excluded

AMZ, 1990
I.A., Medvedev, Zubarev
and PTY, 1990
Preitschopf, Thorn, Yost

Non-polynomial action, Berkovits (1995)

SuperSFT

- Q in “the small Hilbert space” has nontrivial cohomologies
- Q in “the large Hilbert space” has trivial cohomologies (In “the large Hilbert space”)

$$\eta_0, \xi_0 \text{ are exclude}$$

Berkovits a

- Cubic modified action in the large Hilbert space is equivalent to the nonpolinomial action of the WZWN-type

Grassi, Schnabl, 2009-10

$$S_{CS} = -\frac{1}{g^2}\int_{\Sigma\times R^1} Tr[\frac{1}{2}AdA + \frac{1}{3}A^3],$$

Let Σ is an unit disk D

E.Witten, 88

Elitzur,Moor,

Schimmer,

Seiberg, 89

$$A = A_0 + A_r + A_\phi$$

then

$$S_{CS} = -\frac{1}{g^2}\int_{\Sigma\times R^1} Tr[2A_0\tilde{F} - \tilde{A}\partial_t\tilde{A}]dt$$

where

$$d = dt\frac{\partial}{\partial t} + \tilde{d}, \quad \tilde{F} = \tilde{d}\tilde{A} + \tilde{A}\tilde{A}$$

There is a constrain t

$$\tilde{F} = 0$$

$$A_0 |_{\partial\Sigma} = 0$$

$\tilde{F} = 0$ be easily solved, since $\tilde{d}$ has hero hogomology on the disk,

$$\tilde{A} = -\tilde{d}U \cdot U^{-1} \equiv U \cdot \tilde{d}U^{-1}$$

Substituting back to the action we get

$$S_{CS,D}(U) = -\frac{1}{g^2} \int_{S^1 \times R^1} Tr[U^{-1} d_\phi U \cdot U^{-1} d_t U] d\phi \wedge dt - \frac{1}{3g^2} \int_{D \times R^1} Tr[(U^{-1} dU)],$$

Translation to SSFT.

$$\mathbb{Q} = \eta_0 + d_\tau + Q$$
$$\mathbb{A} = A_\eta^{(1,-1,0)} + A_\tau^{(0,0,1)} + A_Q^{(1,0,0)} \equiv B + C + A$$

here $A^{(gh\#,\ pictute\#,\ \#\ of\ dt)}$

The dictionary with previous model

$$d_t \Leftrightarrow \eta_0$$
$$d_r \Leftrightarrow d_\tau$$
$$d_\phi \Leftrightarrow Q$$

The action

$$\mathbb{S} = -\frac{1}{g^2}\langle\langle \frac{1}{2}\mathbb{A}\mathbb{Q}\mathbb{A} + \frac{1}{3}\mathbb{A}^3\rangle\rangle$$

$$\begin{aligned} A_0 &\Rightarrow B \\ \tilde{F} &\Rightarrow (d_\tau + Q)(A + C) + (A + C)^2 = 0 \end{aligned}$$

Cohomologies of $Q + d_\tau$ are trivial

$$H(Q + d_\tau) = 0$$

and

$$A + C = \mathcal{G}^{-1}(Q + dt)\mathcal{G},$$

Result

$$S = \frac{1}{2g^2} < (e^{-\Phi} Q_B e^{\Phi})(e^{-\Phi}\eta_0 e^{\Phi}) < - \int_0^1 (e^{-t\Phi} d_t e^{t\Phi})\{(e^{-t\Phi} Q_B e^{t\Phi}), (e^{-t\Phi}\eta_0 e^{t\Phi})\} >$$

where $\{A, B\} \equiv AB + BA$, and $e^{-t\Phi}\partial_t e^{t\Phi} = \Phi$

Starting point

$$\mathbb{S} = -\frac{1}{g^2}\langle\langle \frac{1}{2}\mathbb{A}\mathbb{Q}\mathbb{A} + \frac{1}{3}\mathbb{A}^3 \rangle\rangle$$

- Cubic modified action in the large Hilbert space is equivalent to the nonpolinomial action of the WZWN-type

$$S_{AMZ-PTY} \simeq S_B$$

Concluding remarks

- **There are many solutions of SFT, the tachyon vacuum being the simplest.**

  Many more remain to be found
- **GSO(-) sector**
- **Ramon sector (analogy with super WZWN model)**
- **Lumps solutions**
- **Time dependent solutions**

In search of new solutions studies of

cohomologies of the BRST operators,

which are closely related with

the **Slavnov**-Taylor identities,

play very important role.

Happy Birth Day!!! !!