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MAXWELL SYMMETRIES AND SUPERSYMMETRIES

Contents:

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1. Maxwell symmetry and particle model in constant EM background

Maxwell algebra = Poincaré algebra $(\mathrm{P}_\mu, \mathrm{M}_{\mu\nu})$ deformed and enlarged by tensorial central charges $\mathrm{Z}_{\mu\nu} \Leftarrow$ Abelian

$$[\mathrm{P}_\mu, \mathrm{P}_\nu] = 0 \rightsquigarrow [\mathrm{P}_\mu, \mathrm{P}_\nu] = i\mathrm{e}\mathrm{Z}_{\mu\nu}$$

$$[\mathrm{P}_\mu, \mathrm{Z}_{\rho\tau}] = [\mathrm{Z}_{\mu\nu}, \mathrm{Z}_{\rho\tau}] = 0 \qquad \leftarrow \text{Abelian}$$

$$[\mathrm{M}_{\mu\nu}, \mathrm{Z}_{\rho\tau}] = i(\eta_{\mu[\nu}\mathrm{Z}_{\nu\rho]} - \eta_{\nu[\rho}\mathrm{Z}_{\mu\tau]})$$

Maxwell algebra by contraction of dS algebra

$$[\mathrm{M}_{\mu\nu}, \mathrm{M}_{\rho\tau}] = -i(\eta_{[\nu\rho}\mathrm{M}_{\nu]\tau} + \eta_{[\mu\tau}\mathrm{M}_{\nu]\rho})$$

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = \frac{i}{R^2}\mathrm{M}_{\mu\nu}$$

We rescale

$$\mathcal{P}_\mu = \frac{1}{\alpha}\mathbf{P}_\mu \qquad \mathbf{M}_{\mu\nu} = \frac{1}{\alpha^2}\mathbf{Z}_{\mu\nu}$$

In the limit one gets $(\mathbf{P}_\mu, \mathbf{Z}_{\mu\nu}) = \mathcal{Z}$ subalgebra of Maxwell algebra $\mathcal{M}$ with $\mathbf{e} = \frac{1}{R^2}$,

$\mathcal{M}$ is obtained by adding the Lorentz generators $\mathbf{M}_{\mu\nu}$ in semisimple way:

$$\mathcal{M} = O(3,1) \niplus \mathcal{Z}$$

Dynamical realization: relativistic particle coupled to EM potential $\mathbf{A}_\mu = \frac{1}{2}\mathbf{f}^{(0)}_{\mu\nu}\mathbf{x}^\nu \leftrightarrow \mathbf{F}_{\mu\nu} = \mathbf{f}^{(0)}_{\mu\nu} \Leftarrow$ constant EM field

$$\mathbf{L} = \mathbf{m}\sqrt{\dot{\mathbf{x}}^2} + \frac{\mathbf{e}}{\mathbf{2}}\mathbf{f}^{(0)}_{\mu\nu}\mathbf{x}^\mu \dot{x}^\nu \leftrightarrow \boldsymbol{\pi}_\mu \dot{\mathbf{x}}^\mu - \frac{\boldsymbol{\lambda}}{\mathbf{2}}(\boldsymbol{\pi}^2 + \mathbf{m}^2) + \frac{\mathbf{e}}{\mathbf{2}}\mathbf{f}^{(0)}_{\mu\nu}\mathbf{x}^\mu \dot{\mathbf{x}}^\nu$$

$$\mathbf{p}_\mu = \boldsymbol{\pi}_\mu - \frac{e}{2}\mathbf{f}^{(0)}_{\mu\nu}\mathbf{x}^\nu \qquad \Rightarrow \{\mathbf{p}_\mu, \mathbf{p}_\nu\} = e\mathbf{f}^{(0)}_{\mu\nu}$$

canonical P.B.

If $f^{(0)}_{\mu\nu} = const. \Rightarrow$ part of Lorentz rotations broken.
In order to recover Maxwell symmetry
$f^{(0)}_{\mu\nu} \Rightarrow f_{\mu\nu}$ - dynamical degree of freedom, i.e. one should extend the phase space as follows ($4+4 \rightarrow 10+10$)

$$(x_\mu, p_\kappa) \longrightarrow (x_\mu, \phi_{\mu\nu}; p_\kappa, f_{\mu\nu}) = (y_A, \widetilde{p}_A)$$

Extended Minkowski space $\Rightarrow$ coset of Maxwell group $\frac{\mathcal{M}}{O(3,1)}$

$$g = e^{iP_\mu x^\kappa}\, e^{\frac{i}{2} Z_{\mu\nu}\, \phi^{\mu\nu}}$$

CM one-form: $\Omega = -ig^{-1}dg = P_\mu e^\mu + \frac{1}{2} Z_{\mu\nu}\omega^{\mu\nu} - \frac{1}{2} M_{\mu\nu} e^{\mu\nu}$

$$e^\mu = dx^\mu \quad \omega^{\mu\nu} = \phi^{\mu\nu} + \frac{1}{2} x^{[\mu} dx^{\nu]} \qquad e^{\mu\nu} = 0$$

$$e^\mu = e^\mu_\tau\, d\tau = \dot{x} d\tau \qquad \omega^{\mu\nu}_\tau = \dot{\phi}^{\mu\nu} + \frac{1}{2} x^{[\mu} \dot{x}^{\nu]}$$

Transition to Maxwell-invariant EM interaction:

$$\frac{1}{2}\mathrm{f}^{(0)}_{\mu\nu}\mathrm{x}^{[\mu}\dot{\mathrm{x}}^{\nu]} \leftarrow \frac{1}{2}\mathrm{f}_{\mu\nu}\,\omega^{\mu\nu}_{\tau} = \frac{1}{2}\mathrm{f}_{\mu\nu}(\dot{\phi}^{\mu\nu} + \mathrm{x}^{[\mu}\dot{\mathrm{x}}^{\nu]})$$

E-L equations of motion

$$m\ddot{\mathrm{x}}_{\mu} = \mathrm{f}_{\mu\nu}\dot{\mathrm{x}}^{\nu} \qquad \dot{\phi}^{\mu\nu} = -\mathrm{x}^{[\mu}\dot{\mathrm{x}}^{\nu]} \qquad \dot{\mathrm{f}}_{\mu\nu} = 0$$

Generalized EM potential one-form $\rightarrow$ field strength two-form:

$$\mathcal{A} = \tfrac{1}{2}\mathrm{f}_{\mu\nu}\omega^{\mu\nu} \longrightarrow \mathcal{F} = d\mathcal{A} = \underset{\substack{\Uparrow \\ \text{constant on-shell}}}{\tfrac{1}{2}\mathrm{f}_{\mu\nu}}e^{\mu}\wedge e^{\nu} + \underset{\substack{\shortparallel \\ 0 \\ \text{on shell}}}{\tfrac{1}{2}d\mathrm{f}_{\mu\nu}}\wedge\omega^{\mu\nu}$$

Realization of Maxwell algebra:

$$\mathrm{P}_{\mu} = -(\mathrm{p}_{\mu} - \frac{1}{2}\mathrm{f}_{\mu\nu}\mathrm{x}^{\nu}) \qquad \mathrm{M}_{\mu\nu} = -(\mathrm{p}_{[\mu}\mathrm{x}_{\nu]} + \mathrm{f}_{[\mu\rho}\phi^{\rho}_{\ \nu]})$$

$$\mathrm{Z}_{\mu\nu} = -\mathrm{f}_{\mu\nu}$$

Constraints:

$$\mathrm{C}_1 = \phi = \pi_\mu^2 + \mathrm{m}^2 = 0 \qquad \pi_\mu = \mathrm{p}_\mu + \frac{1}{2}\mathrm{f}_{\mu\nu}\mathrm{x}^\nu$$

C_1 describes the mass Casimir of Maxwell algebra, besides there are **three additional Casimirs**

$$C_2 - \frac{1}{2}\mathrm{f}_{\mu\nu}\mathrm{f}^{\mu\nu} \qquad C_3 = \frac{1}{2}\varepsilon^{\mu\nu\rho\tau}\mathrm{f}_{\mu\nu}\mathrm{f}_{\rho\tau} \qquad C_4 = (\pi_\mu \mathrm{f}^{\mu\nu})^2$$

Quantization: extended KG equation

$$[(\frac{1}{i}\frac{\partial}{\partial \mathrm{x}^\mu} + \frac{1}{2i}\mathrm{x}^\nu \frac{\partial}{\partial \phi^{\mu\nu}})^2 + m^2]\psi(\mathrm{x}^\mu, \phi^{\mu\nu}) = 0$$

$$\updownarrow \quad F.T. \quad \updownarrow$$

$$[(\frac{1}{i}\frac{\partial}{\partial \mathrm{x}^\mu} - \frac{1}{2}\mathrm{x}^\nu \mathrm{f}_{\mu\nu})^2 + m^2]\widetilde{\psi}(\mathrm{x}^\mu, \mathrm{f}_{\mu\nu}) = 0$$

Remaining three Casimirs: C_2, C_3 specifies $f_{\mu\nu}$
C_4 - second order diff. Eq.

Assuming that only $f = f_{12} \neq 0$ one gets from C_1 KG operator with added

- oscillator - like term $\sim f^2(x_1^2 + x_2^2)$
- angular momentum term $\sim f J_3$ $\qquad J_3 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$

Interestingly, similar two terms appear if we write the harmonic oscillator in noncommutative space $\widehat{x}_i$ with

$$[\widehat{x}_i, \widehat{x}_j] = i\theta\varepsilon_{ij} \qquad \left(\begin{array}{c}\text{special case of} \\ [\widehat{x}_\mu, \widehat{x}_\nu] = i\theta_{\mu\nu}\end{array}\right)$$

because one can express $\widehat{x}_i$ by commutative x_i

$$\widehat{x}_i = x_i - \frac{1}{2}\theta\varepsilon_{ij} p_j$$

$\Rightarrow$ links between dynamics in noncommutative space-time and in noncommutative momentum space

2. Deformations of Maxwell Symmetries

i) Arbitrary D - one-parameter deformation

1) $k > 0$ (k - **deformation parameter)**

$$\text{Maxwell algebra} \longrightarrow \underset{\substack{\text{D-dimensional} \\ \text{AdS}}}{SO(D-1,2)} \oplus \underset{\substack{\text{D-dimensional} \\ \text{Lorentz}}}{SO(D-1,1)}$$

2) $k < 0$

$$\text{Maxwell algebra} \longrightarrow \underset{\text{dS}}{SO(D,1)} \oplus \underset{\text{Lorentz}}{SO(D-1,1)}$$

Remark: **the case** $k > 0$ **was firstly given by Soroka and Soroka in 2006**

ii) $D = 2+1$ **- two deformation parameters** k, b

$$[P_a, P_b] = -i\varepsilon_{abc}\mathrm{Z}^c \qquad [\mathrm{M}_a, \mathrm{M}_b] = i\varepsilon_{abc}\mathrm{M}^c \quad (a = 0,1,2)$$

$$[M_b, P_a] = -i\varepsilon_{abc}\mathrm{P}^c \qquad [M_a, Z_b] = -i\varepsilon_{abc}\mathrm{Z}^c$$

$$[P_a, Z_b] = -ik\varepsilon_{abc}\mathrm{P}^c - ib\varepsilon_{abc}\mathrm{M}^c$$

$$[Z_a, Z_b] = -ik\varepsilon_{abc}\mathrm{Z}^c - ib\varepsilon_{abc}\mathrm{P}^c$$

$b = 0$ **: fourlinear relation for** P_a

$$g_{ij} = C^l_{ik} C^k_{lj} \Rightarrow \det \sim (\frac{k}{3})^3 - (\frac{b}{2})^2 \qquad (b \sim \frac{1}{R^3})$$

Three sectors:

$\det g > 0$	$SO(2,2) \oplus SO(2,1)$	$(AdS \oplus \text{Lorentz})$
$\det g < 0$	$SO(3,1) \oplus SO(2,1)$	$(dS \oplus \text{Lorentz})$
$\det g = 0$	$SO(2,1) \oplus SO(2,1)$	$(\text{Poincaré} \oplus \text{Lorentz})$

Realization via particle model + EM coupling: k-extended

Maxwell-invariant particle action: $(D = 4, k > 0)$

$$\mathcal{L}d\tau = -m\sqrt{-\eta_{\mu\nu}e^{\mu}_{\tau}e^{\nu}_{\tau}} + \tfrac{1}{2}\mathrm{f}_{\mu\nu}\omega^{\mu\nu}_{\tau} = -m\sqrt{-g_{\mu\nu}(\mathrm{x})\dot{\mathrm{x}}^{\mu}\mathrm{x}^{\nu}} + \mathcal{A}$$

$\uparrow$ P_μ − one-form $\qquad$ $\nwarrow$ $Z_{\mu\nu}$ − one-form

$$g_{\mu\nu}(\mathrm{x}) = \eta_{\mu\nu} + [(\frac{sin\sqrt{k'r^2}}{\sqrt{kr^2}})^2 - 1](\eta_{\mu\nu} - \frac{\mathrm{x}_\mu \mathrm{x}_\nu}{\mathrm{x}^2})$$

***AdS* metric**

Field equations:

$\swarrow$ generalized Lorentz force

$$m\nabla_\tau \dot{\mathrm{x}}_\mu = F_{\mu\nu}\dot{\mathrm{x}}^\nu \qquad \nabla_\tau \dot{\mathrm{x}}_\mu \sim g_{\mu\nu}(\dot{\mathrm{x}}^\nu + \Gamma^\nu_{\rho\tau}\dot{\mathrm{x}}^\rho \dot{\mathrm{x}}^\tau)$$

$\Uparrow$ AdS connection

$$\mathfrak{F} = d\mathcal{A} = \tfrac{1}{2}F_{\mu\nu}d\mathrm{x}^\mu \wedge d\mathrm{x}^\nu \leftarrow \text{on-shell}$$

$$F_{\mu\nu} = \widetilde{f}_{\mu\nu}(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}})^2 - \frac{\widetilde{\mathrm{f}}_{[\mu\rho}\mathrm{x}^\rho \mathrm{x}_{\nu]}}{\mathrm{x}^2}\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}}(\sin\frac{\sqrt{kr^2}}{\sqrt{kr^2}} - 1)$$

$$\widetilde{f}_{\mu\nu} = (e^{-k\phi})_\mu^{\ \rho} f_{\rho\tau}(e^{k\phi})^\tau_{\ \nu}$$

On-shell: $\phi_{\mu\nu} = \phi_{\mu\nu}(\mathrm{x})$ (nonlocal solutions)

$\dot{f}_{\rho\tau} = 0$ (constant f)

3. Maxwell N=1 Superalgebra

SUSY strength superfield:

Constant EM field $f_{\mu\nu}$ $\longrightarrow$ $W_\alpha(\theta) = i\lambda_\alpha - \frac{i}{2} f_{\mu\nu}(\overline{\theta}\gamma^{\mu\nu})_\alpha - iD(\overline{\theta}\gamma_5)_\alpha$

$\Uparrow$

constant Abelian N=1

Minimal extension: $Q_\alpha \longrightarrow (Q_\alpha, \Sigma_\alpha)$

$$\{Q_\alpha, Q_\beta\} = 2(C\gamma^4)_{\alpha\beta} P_\mu \qquad [P_\mu, P_\nu] = iZ_{\mu\nu}$$

$$\Downarrow$$

$$\{Q_\alpha, \Sigma_\beta\} = \frac{1}{2}(C\gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} \qquad [P_\mu, Q_\alpha] = i\Sigma_\beta(\gamma_\mu)_\alpha^\beta$$

$$\{\Sigma_\alpha, \Sigma_\beta\} = 0 \qquad [P_\mu, \Sigma_\alpha] = 0$$

SUSY extension of mass Casimir:

$$\widetilde{C}_1 = \underbrace{P_\mu P^\mu + M_{\mu\nu} Z^{\mu\nu}}_{C_1} - i\Sigma_\alpha C^{\alpha\beta} Q_\beta$$

One can add **scalar central charge B:**

$$\{Q_\alpha, \Sigma_\beta\} = \frac{1}{2}(C^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (C\gamma_\sigma)_{\alpha\beta} B$$

Generators: $\underbrace{(P_\mu, M_{\mu\nu}, Q_\alpha}_{\text{N=1}}, \; Z_{\mu\nu}, \; \Sigma_\mu, \; B)$

$$\begin{array}{cccc} & \uparrow & \uparrow & \uparrow \\ \text{superPoincaré} & f_{\mu\nu} & \lambda_\alpha & D \end{array}$$

In order to obtain supersymmetrized Casimir $\widetilde{C}_1$ it is necessary to add **second "axial" scalar generator B_5**

$$[B_5, Q_\alpha] = -i(Q\gamma_5)_\alpha \qquad [B_5, \Sigma_\alpha] = i(\Sigma\gamma_5)_\alpha$$

and one gets

$$\widetilde{\widetilde{C}}_1 = \widetilde{C}_1 - B_5 \cdot B$$

4. Maxwell superparticle model with Maxwell supersymmetry

$$\tilde{g} = \frac{\widetilde{\mathcal{M}}_B}{SO(3,1)} = e^{\frac{i}{2}Z_{\mu\nu}\phi^{\mu\nu}}\, e^{iP_\mu \mathrm{x}^\mu}\, e^{i\Sigma_\alpha \phi^\alpha}\, e^{iQ_\alpha \theta^\alpha}\, e^{iB\phi}$$

CM one-forms: $\Omega = \tilde{e}^\mu P_\mu + \tilde{\omega}^{\mu\nu} Z_{\mu\nu} + \tilde{\omega}^\alpha \Sigma_\alpha + \tilde{\omega}\, B + \ldots$

$$\tilde{e}^\mu = dx^\mu + i(\bar{\theta}\gamma^\mu d\theta) \quad \tilde{\omega}^\alpha = d\phi^\alpha + (\gamma_\mu \theta)^\alpha (dx^\mu + \tfrac{i}{3}(\bar{\theta}\gamma^\rho d\theta))$$

$$\tilde{\omega}^{\mu\nu} = d\phi^{\mu\nu} + \tfrac{1}{2}x^{[\mu}dx^{\nu]} + i(\bar{\theta}\gamma^{\mu\nu}d\theta) + \tfrac{i}{2}(\bar{\theta}\gamma^{\mu\nu}\gamma_\rho\theta)(dx^4 + \tfrac{i}{6}(\bar{\theta}\gamma^\rho d\theta))$$

$$\tilde{\omega} = d\phi + i(\bar{\theta}\gamma_5 d\phi) + \tfrac{i}{2}(\bar{\theta}\gamma_5\theta)(dx^\rho + \tfrac{i}{6}(\bar{\theta}\gamma^\rho d\theta))$$

$\ldots$

Superparticle action (massless):

$$L = \frac{\pi_\mu \pi^\mu}{2e} + \tfrac{1}{2} f_{\mu\nu}\tilde{\omega}^{\mu\nu}_\tau + i\lambda_\alpha \tilde{\omega}^\alpha_\tau + D\tilde{\omega}_\tau = L^0 + L_{\text{int}}$$

$$\Uparrow$$

$\pi_\mu \equiv \tilde{e}_{\tau\mu} = \dot{x}_\mu + i\bar{\theta}\gamma_\mu\dot{\theta}$ − pullback of $\tilde{e}_\mu$

Superparticle equation of motions (variations $\delta x_\mu, \delta\theta_\alpha$)

$$\frac{d}{d\tau}\left(\frac{\pi_\mu}{e}\right) = \pi^\nu F_{\mu\nu} + \dot{\theta}^\beta F_{\mu\beta}$$
$$2i(\dot{\bar{\theta}}\gamma^\mu)_\alpha\left(\frac{\pi_\mu}{e}\right) = \pi^\nu F_{\nu\alpha}$$

and

$$\pi_\mu \pi^\mu = 0 \qquad \text{(mass-shell condition)}$$

The model is invariant under

i) superMaxwell invariance

ii) τ-reparametrizations

(iii) κ-symmetries (because $F_{\mu\nu}, F_{\nu\alpha}$ satisfies the superspace constraints for SUSY gauge theory)

Remaining degrees of freedom $\phi^{\mu\nu}, \phi^\alpha, \phi$ - on-shell can be expressed (nonlocally) by the trajectories $x^\mu(\tau), \theta^\alpha(\tau)$

5. Outlook

a) One can consider the deformation of Maxwell superalgebra $\widetilde{\mathcal{M}}$. One can deform as follows (for D=4)

$$\widetilde{\mathcal{M}} \begin{array}{l} \nearrow \boldsymbol{OSp(2|4)} \\ \\ \searrow \boldsymbol{O^{*}Sp(2|4) \simeq UU_{\alpha}(1,1|1;H)} \end{array}$$

i.e. one can also consider $\mathcal{M}$ as contraction of $\boldsymbol{OSp(2|4)}$ or $\boldsymbol{UU_{\alpha}(1,1|1;H)}$ (Bonanos, Gomis, Kamimura + J.L., in preparation)

b) Possible dynamical application: physics in the presence of constant background $\leftrightarrow$ uniform filling of space-time linked with some interpretations of dark energy and dynamical appearance of cosmological constant parameter.