FROM SLAVNOV–TAYLOR IDENTITIES TO THE ZJ EQUATION

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In our work on proving the renormalizability of non-Abelian gauge theories in the broken phase, we (Lee and Zinn-Justin) have directly benefited from Slavnov's important contributions, as I shall recall here.

As a general reference see

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, 1989) International Series of Monographs on Physics 113, 1054 pp. (2002), Fourth Edition.

See also

Andrei A. Slavnov, *Slavnov–Taylor identities,* (2008) Scholarpedia, 3(10): 7119.

Jean Zinn-Justin, *Zinn-Justin equation,* (2009) Scholarpedia, 4(1): 7120.

ST identities: The origin

Slavnov identities in gauge theories owe less to gauge symmetry than to gauge fixing, as we first show.

Let φ^α be a set of dynamical variables satisfying a system of equations,

$$E_\alpha(\varphi) = 0\,,$$

where the functions $E_\alpha(\varphi)$ are smooth, and $E_\alpha = E_\alpha(\varphi)$ is a one-to-one mapping in some neighbourhood of $E_\alpha = 0$, which can be inverted in $\varphi^\alpha = \varphi^\alpha(E)$. In particular, this implies that the equation has a unique solution $\varphi^\alpha_{\mathrm{s}} \equiv \varphi^\alpha(0)$. In the neighbourhood of φ_{s}, the determinant $\det \mathbf{E}$ of the matrix $\mathbf{E}$ with elements

$$E_{\alpha\beta} \equiv \partial_\beta E_\alpha\,,$$

does not vanish and thus one can choose $E_\alpha(\varphi)$ such that it is positive.

For any function $F(\varphi)$, one can derive a simple formal expression for $F(\varphi_{\mathrm{s}})$, which does not involve solving the equation explicitly. One starts from the trivial identity

$$F(\varphi_{\mathrm{s}}) = \int \left\{ \prod_{\alpha} \mathrm{d}E^{\alpha} \, \delta(E_{\alpha}) \right\} F\big(\varphi(E)\big),$$

where $\delta(E)$ is Dirac's δ-function. One then changes variables $E \mapsto \varphi$. This generates the Jacobian $\det \mathbf{E} > 0$. Thus,

$$F(\varphi_{\mathrm{s}}) = \int \left\{ \prod_{\alpha} \mathrm{d}\varphi^{\alpha} \, \delta\left[E_{\alpha}(\varphi)\right] \right\} \det \mathbf{E}(\varphi) \, F(\varphi). \tag{1}$$

In the context of non-Abelian gauge theories, $\det \mathbf{E}$ is the Faddeev–Popov determinant.

An invariant measure

$\prod_\alpha \mathrm{d}E_\alpha$ is the invariant measure for the group of translations $E_\alpha \mapsto E_\alpha + \nu_\alpha$. It follows that the measure

$$\mathrm{d}\rho(\varphi) = \det \mathbf{E}(\varphi) \prod_\alpha \mathrm{d}\varphi^\alpha \,, \tag{2}$$

is the invariant measure for the translation group realized non-linearly on the new coordinates φ_α (provided ν_α is small enough):

$$\varphi^\alpha \mapsto \varphi'^\alpha \quad \text{with} \quad E_\alpha(\varphi') - \nu_\alpha = E_\alpha(\varphi).$$

The infinitesimal form of the transformation can be written more explicitly as

$$\delta\varphi^\alpha = [E^{-1}(\varphi)]^{\alpha\beta}\nu_\beta \,. \tag{3}$$

This rather straightforward property of the measure, and the corresponding infinitesimal transformations (3), have been used in the context of non-Abelian gauge theories by Slavnov and, independently Taylor, to derive a set of important identities satisfied by Green's functions thus called Slavnov–Taylor identities.

These identities form the basis of the first proof by Lee and Zinn-Justin of the renormalizability of non-Abelian gauge theories in the broken phase. This method has then been extended to various other field theories, for example, to the dynamics generated by a Langevin equation.

Reciprocal property. Conversely, one can characterize the general form of non-linear representations of the translation group. One recovers the form of the previous measure.

From ST symmetry to BRST symmetry

In quantum field theory, the non-linear and non-local character of the transformations (3) is the source of some technical complications. Remarkably enough, the invariance under the infinitesimal transformations (3) can be replaced by an invariance under linear anticommuting-type transformations at the price of introducing additional Grassmann variables.

One again starts from the identity (1) and first replaces the δ-function by its Fourier representation:

$$\prod_\alpha \delta\left[E_\alpha(\varphi)\right] = \int \prod_\alpha \frac{\mathrm{d}\bar\varphi^\alpha}{2i\pi}\,\mathrm{e}^{-\bar\varphi^\alpha E_\alpha(\varphi)},$$

where the $\bar\varphi$ integration runs along the imaginary axis. Moreover, a determinant can be written as an integral over Grassmann variables $\bar c^\alpha$ and c^α:

$$\det \mathbf{E} = \int \prod_\alpha \left(\mathrm{d}\bar c^\alpha \mathrm{d}c^\alpha\right) \exp\left(c^\alpha E_{\alpha\beta}\bar c^\beta\right).$$

The expression (1) then becomes

$$F(\varphi_{\mathrm{s}}) = \mathcal{N} \int \prod_{\alpha} (\mathrm{d}\varphi^{\alpha} \mathrm{d}\bar{\varphi}^{\alpha} \mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha})\, F(\varphi) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c})\right],$$

in which $\mathcal{N}$ is a constant normalization factor and $S(\varphi, \bar{\varphi}, c, \bar{c})$ the function (and element of the Grassmann algebra)

$$S(\varphi, \bar{\varphi}, c, \bar{c}) = \bar{\varphi}^{\alpha} E_{\alpha}(\varphi) - c^{\alpha} E_{\alpha\beta}(\varphi) \bar{c}^{\beta}\,. \tag{4}$$

Somewhat surprisingly, the function S has a new type of symmetry, which is directly related to the invariance of the measure $\det \mathbf{E}(\varphi) \mathrm{d}\boldsymbol{\varphi}$ under the group of transformations $\delta\varphi^{\alpha} = [E^{-1}(\varphi)]^{\alpha\beta} \nu_{\beta}$.

In the context of non-Abelian gauge theories, $\mathbf{c}$ and $\bar{\mathbf{c}}$ are the Faddeev–Popov ghosts.

BRST symmetry

The BRST symmetry, first discovered in the context of quantized gauge theories by Becchi, Rouet, Stora, and Tyutin, is a Grassmann symmetry in the sense that the parameter ε of the transformation is an anticommuting constant, an additional generator of the Grassmann algebra. The variations of the various dynamic variables are

$$
\begin{aligned}
\delta\varphi^\alpha &= \varepsilon\bar{c}^\alpha\,, &\qquad \delta\bar{c}^\alpha &= 0\,,\\
\delta c^\alpha &= \varepsilon\bar{\varphi}^\alpha\,, &\qquad \delta\bar{\varphi}^\alpha &= 0
\end{aligned}
$$

with

$$
\varepsilon^2 = 0\,, \qquad \varepsilon\bar{c}^\alpha + \bar{c}^\alpha\varepsilon = 0\,, \qquad \varepsilon c^\alpha + \varepsilon c^\alpha = 0\,.
$$

The transformation is obviously *nilpotent* of vanishing square: $\delta^2 = 0$.

BRST symmetry and group manifolds

When the variables φ^α parametrize an element $\mathbf{g}(\varphi)$ of a Lie group in some matrix representation, it is convenient to express BRST transformations on $\mathbf{g}(\varphi)$ directly and to parametrize the variation of $\mathbf{g}$ in terms of a Grassmann matrix $\mathbf{C}$ belonging to the Lie algebra of the group:

$$\delta\mathbf{g} = \varepsilon\bar{\mathbf{C}}\mathbf{g}\,.$$

Thus,

$$\bar{\mathbf{C}} = \bar{\mathbf{c}}_\alpha \frac{\partial\mathbf{g}}{\partial\varphi_\alpha}\mathbf{g}^{-1}.$$

The calculation of the variation of $\mathbf{C}$ then yields

$$\delta\bar{\mathbf{C}} = -\bar{\mathbf{c}}_\alpha \frac{\partial\mathbf{g}}{\partial\varphi_\alpha}\mathbf{g}^{-1}\varepsilon\bar{\mathbf{c}}_\beta \frac{\partial\mathbf{g}}{\partial\varphi_\beta}\mathbf{g}^{-1} = \varepsilon\mathbf{C}^2,$$

which is the expression that appears in gauge theories, the group element being there associated with gauge transformations.

BRST transformations: differential operator representation

The BRST transformation, when acting on functions of $\{\varphi, \bar{\varphi}, c, \bar{c}\}$, can be represented by the Grassmann differential operator

$$\mathcal{D} \equiv \bar{c}^{\alpha} \frac{\partial}{\partial \varphi^{\alpha}} + \bar{\varphi}^{\alpha} \frac{\partial}{\partial c^{\alpha}} \,.$$

The nilpotency of the BRST transformation is then expressed by the identity

$$\mathcal{D}^2 = 0 \,.$$

The differential operator $\mathcal{D}$ has the form of a cohomology operator, generalization of the exterior differentiation of differential forms. In particular, the first term $\bar{c}^{\alpha}\partial/\partial\varphi_{\alpha}$ in the BRST operator is identical to the differentiation of forms in a formalism in which the Grassmann variables $\bar{c}^{\alpha}$ are introduced to represent forms.

The equation $\mathcal{D}^2 = 0$ implies that all quantities of the form $\mathcal{D}Q(\varphi, \bar{\varphi}, c, \bar{c})$, quantities one calls BRST exact, are BRST invariant. One immediately verifies that the function S defined by equation (4) is BRST exact:

$$S = \mathcal{D}\left[c^\alpha E_\alpha(\varphi)\right].$$

It follows that S is BRST invariant,

$$\mathcal{D}S = 0\,.$$

The reciprocal property, the meaning and implications of the BRST symmetry rely on simple considerations of BRST cohomology.

These properties play an important role, in particular, in the discussion of the renormalization of non-Abelian gauge theories.

From BRST symmetry to the ZJ equation

In gauge theories, the BRST transformations take the explicit form (we omit matter fields for simplicity)

$$\begin{cases} \delta \mathbf{A}_\mu(x) = -\varepsilon \mathbf{D}_\mu \bar{\mathbf{C}}(x), & \delta \bar{\mathbf{C}}(x) = \varepsilon \bar{\mathbf{C}}^2(x), \\ \delta \mathbf{C}(x) = \varepsilon \boldsymbol{\lambda}(x), & \delta \boldsymbol{\lambda}(x) = 0\,. \end{cases} \tag{5}$$

From the viewpoint of renormalization, the important remark is that the BRST variations are quadratic in the fields and are, therefore, renormalized: the explicit form of the BRST transformations is not stable under renormalization.

Stable equations involve adding to the quantized action $\mathcal{S}$ sources $\mathbf{K}_\mu$ and $\mathbf{L}$ for the non-trivial BRST transformations :

$$\mathcal{S}(\mathbf{A}_\mu, \mathbf{C}, \bar{\mathbf{C}}, \boldsymbol{\lambda}) \mapsto \mathcal{S}(\mathbf{A}_\mu, \mathbf{C}, \bar{\mathbf{C}}, \boldsymbol{\lambda}) + \int \mathrm{d}^4 x \, \mathrm{tr}\left(-\mathbf{K}_\mu(x) \mathbf{D}_\mu \mathbf{C}(x) + \mathbf{L}(x) \mathbf{C}^2(x)\right).$$

The sources $\mathbf{K}_\mu$ and $\mathbf{L}$ for BRST transformations have been later named anti-fields. No other terms are required because the two composite operators in the BRST transformations are BRST invariant.

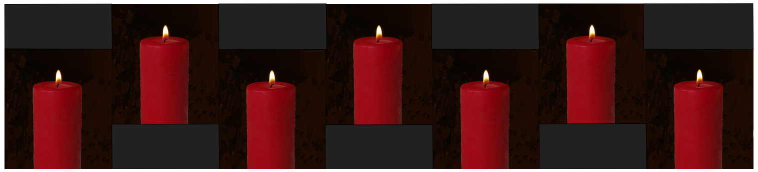
It can be verified (ZJ 1974) that that the complete action

$$\mathcal{S}(\mathbf{A}_\mu, \mathbf{C}, \bar{\mathbf{C}}, \boldsymbol{\lambda}, \mathbf{K}_\mu, \mathbf{L}),$$

which includes these additional source terms, satisfies after renormalization a quadratic relation, the ZJ equation, which does not involve the explicit form of the BRST transformations (5). In component form, the equation reads

$$\int d^4x \sum_\alpha \left(\frac{\delta \mathcal{S}}{\delta A_\mu^\alpha(x)} \frac{\delta \mathcal{S}}{\delta K_\mu^\alpha(x)} + \frac{\delta \mathcal{S}}{\delta C^\alpha(x)} \frac{\delta \mathcal{S}}{\delta L^\alpha(x)} + \lambda^\alpha(x) \frac{\delta \mathcal{S}}{\delta \bar{C}^\alpha(x)} \right) = 0\,.$$

Remarkably enough, the general solution of this equation, taking into account locality, power counting and ghost number conservation, takes the form of a BRST invariant gauge action.



HAPPY BIRTHDAY ANDREI, BEST WISHES for the next decade,

and still many years of productive scientific life.