AdS/CFT integrability in diverse dimensions

Konstantin Zarembo

École Normale Supérieure, Paris
&
ИТЭФ

“Gauge Fields. Yesterday, Today, Tomorrow”, Moscow, 21.01.2010

А. А. СЛАВНОВ, Л. Д. ФАДДЕЕВ

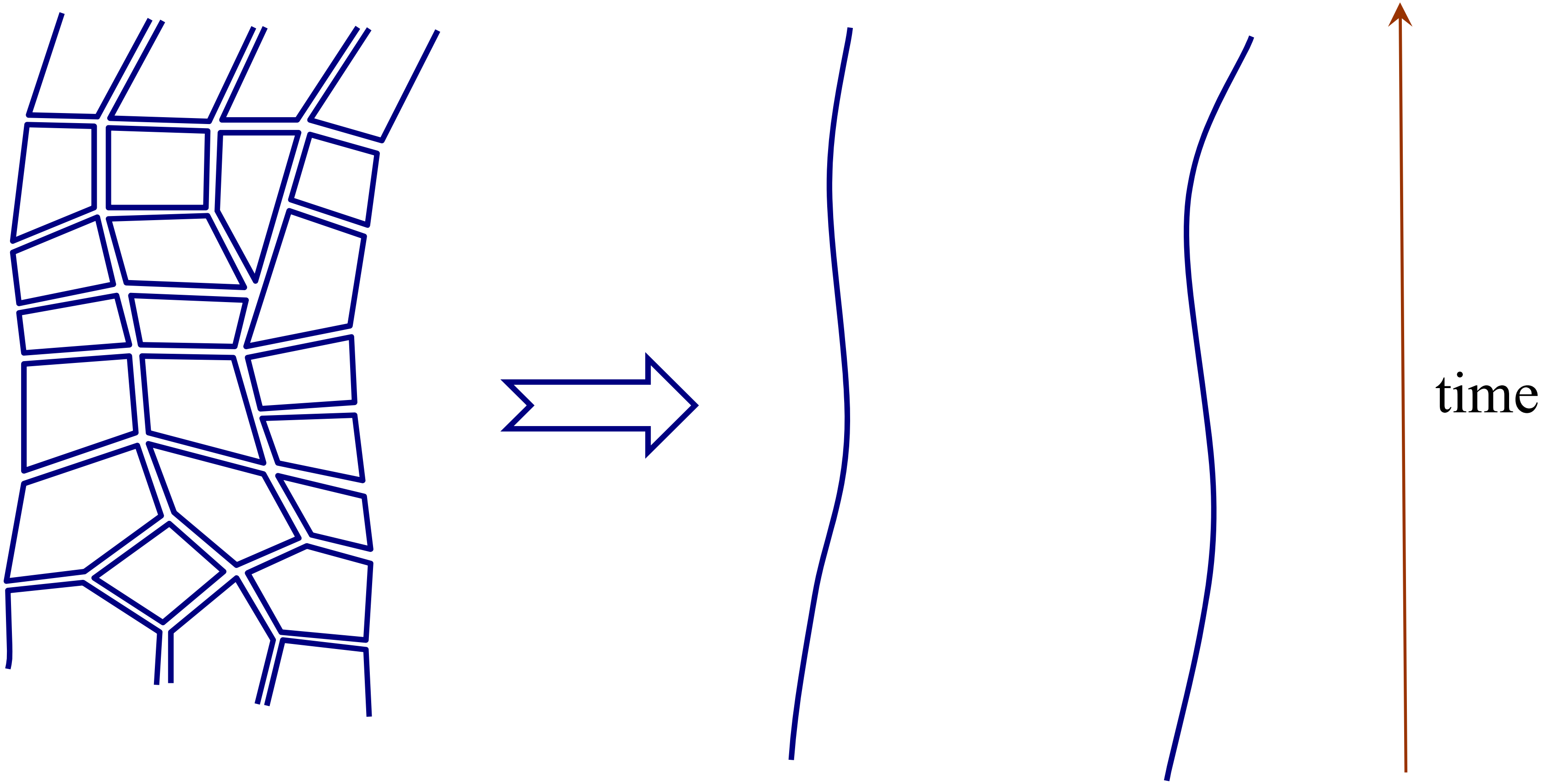
ВВЕДЕНИЕ В КВАНТОВУЮ ТЕОРИЮ КАЛИБРОВОЧНЫХ ПОЛЕЙ

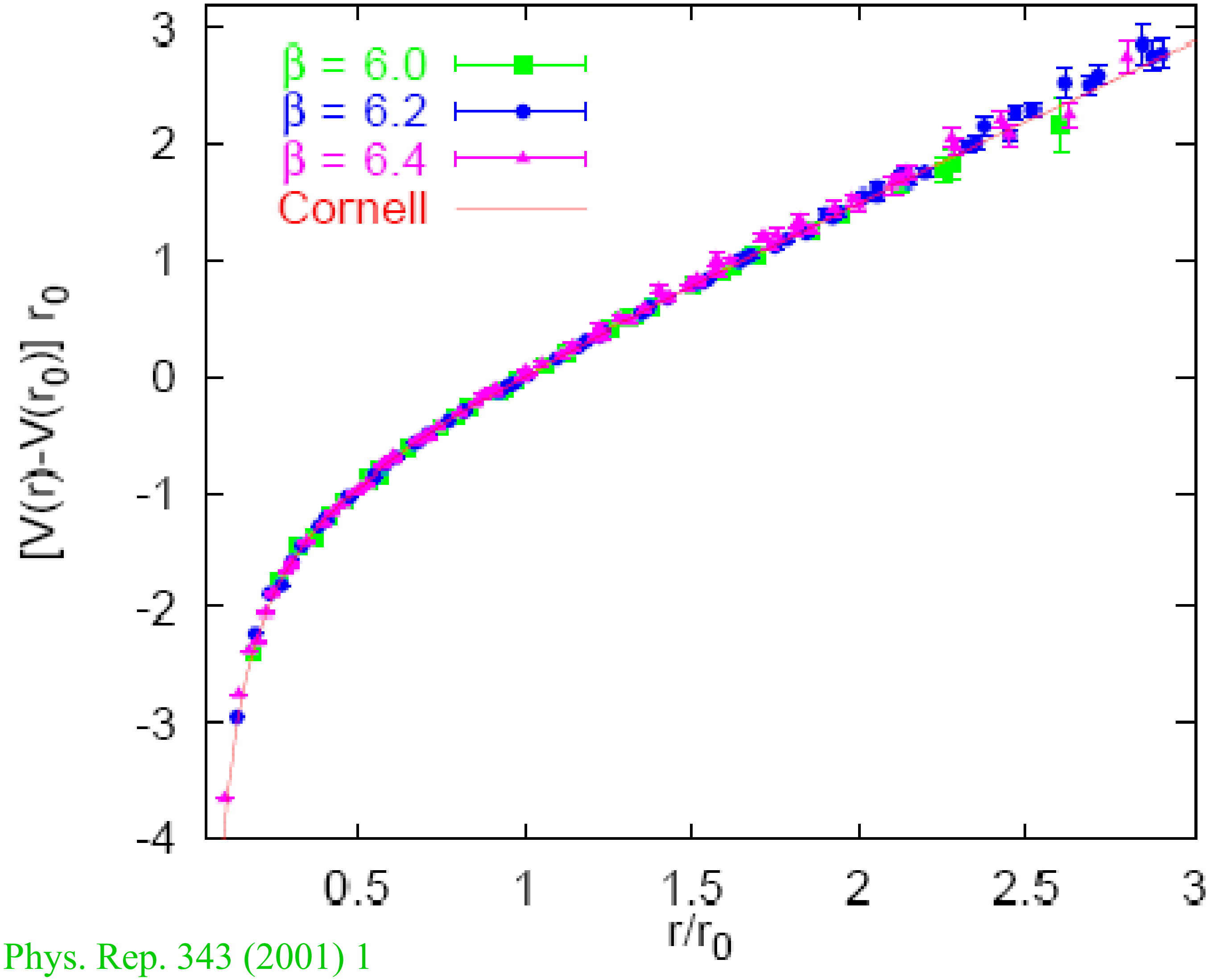
ИЗДАНИЕ ВТОРОЕ,
ПЕРЕРАБОТАННОЕ И ДОПОЛНЕННОЕ

МОСКВА «НАУКА»
ГЛАВНАЯ РЕДАКЦИЯ
ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ
1988

Planar diagrams and strings

Large-N limit: $N \to \infty, \qquad \lambda = g_{\rm YM}^2 N - \text{fixed}$





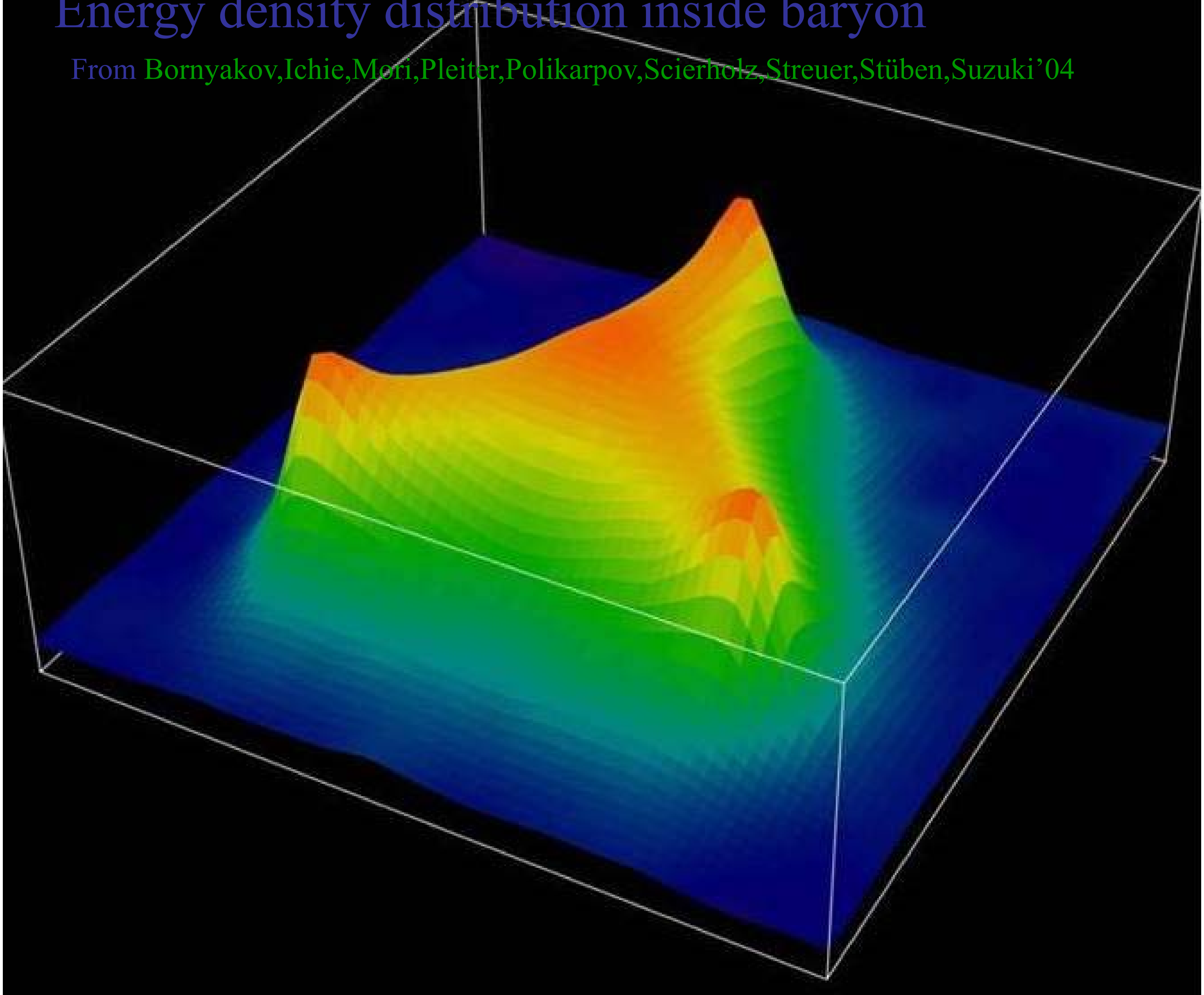
From Bali, Phys. Rep. 343 (2001) 1

More refined lattice simulations confirm that the string fluctuates.

Caselle,Fiore,Gliozzi,Hasenbusch,Provero’97; Caselle,Pepe,Rago’04; …

Energy density distribution inside baryon

From Bornyakov,Ichie,Mori,Pleiter,Polikarpov,Scierholz,Streuer,Stüben,Suzuki’04



AdS/CFT correspondence

Yang-Mills theory with
N=4 supersymmetry

Exact equivalence ⇕

Maldacena’97
Gubser,Klebanov,Polyakov’98
Witten’98

String theory on
$AdS_5 \times S^5$ background

Anti-de-Sitter space (AdS$_5$)

$$\boxed{ds^2 = \frac{dx^\mu dx_\mu + dz^2}{z^2}}$$

z

5D bulk

strings

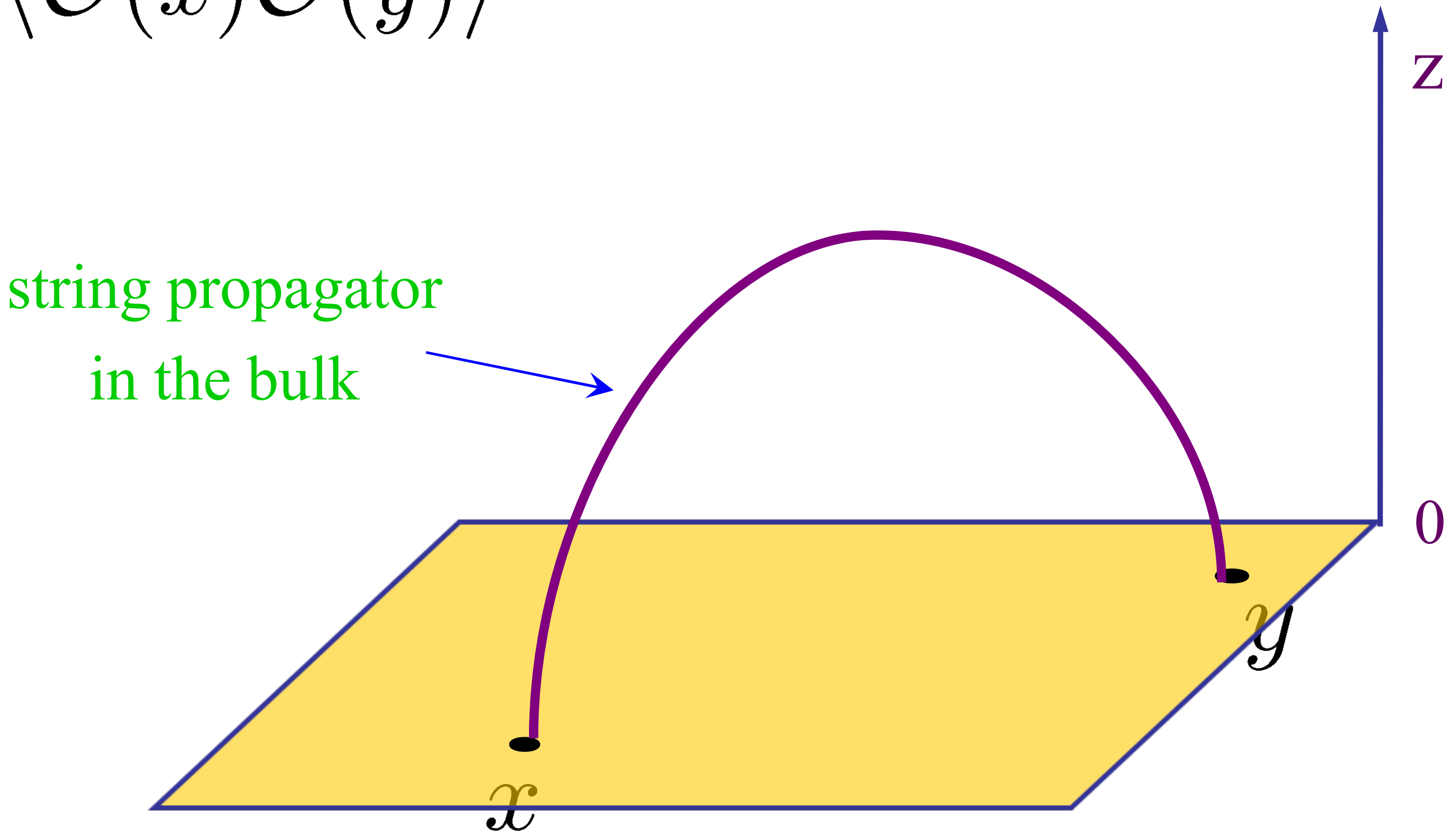
0

gauge fields

4D boundary

Two-point correlation functions

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$$



string propagator in the bulk

AdS/CFT correspondence

Maldacena'97

$\mathcal{N} = 4$ SYM	Strings on $AdS_5 \times S^5$
't Hooft coupling: $\lambda = g_{YM}^2 N$	String tension: $T = \frac{\sqrt{\lambda}}{2\pi}$
Number of colors: N	String coupling: $g_s = \frac{\lambda}{4\pi N}$
Large-N limit	Free strings
Strong coupling	Classical strings
Local operators Scaling dimension: Δ	String states Energy: E

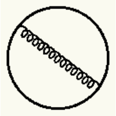
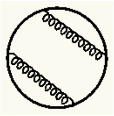
Gubser,Klebanov,Polyakov'98
Witten'98

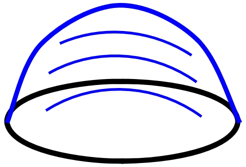
Strong-weak coupling interpolation

0 λ

SYM perturbation theory

String perturbation theory

$1+$  $+$  $+\dots$



Circular Wilson loop (exact):

$$W(\text{circle})=\frac{2}{\sqrt{\lambda}}\,I_1\left(\sqrt{\lambda}\right)$$

Erickson,Semenoff,Z.'00

Drukker,Gross'00

Pestun'07

$\lambda\to 0$

$\lambda\to\infty$

$$W(\text{circle})=1+\frac{\lambda}{8}+\frac{\lambda^2}{192}+\dots$$

$$W(\text{circle})=\sqrt{\frac{2}{\pi}}\lambda^{-3/4}\,\mathrm{e}^{\sqrt{\lambda}}$$

Minimal area law in AdS_5

Correlation functions

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = \frac{1}{|x-y|^{2\Delta}}$$

Dilatation operator:

$$\hat{D}\, |\mathcal{O}\rangle = \Delta\, |\mathcal{O}\rangle$$

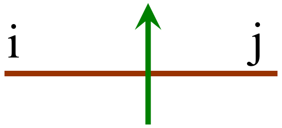
$$\hat{D} = \Delta_0 + \underbrace{\lambda \hat{D}_1 + \lambda^2 \hat{D}_2 + \lambda^3 \hat{D}_3 + \ldots}_{\text{matrix of anomalous dimensions}}$$

Local operators and spin chains

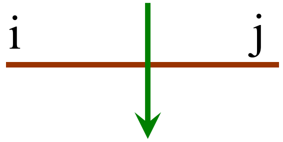
$$Z = \Phi_1 + i\Phi_2$$

$$W = \Phi_3 + i\Phi_4$$

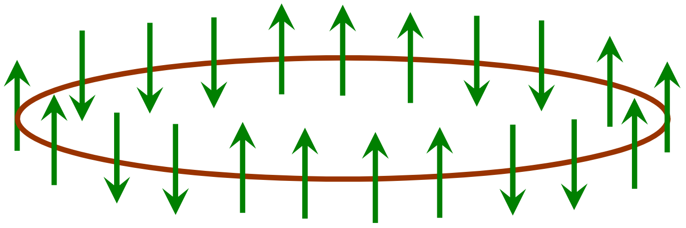
$$Z_{ij} \quad \Longleftrightarrow$$



$$W_{ij} \quad \Longleftrightarrow$$

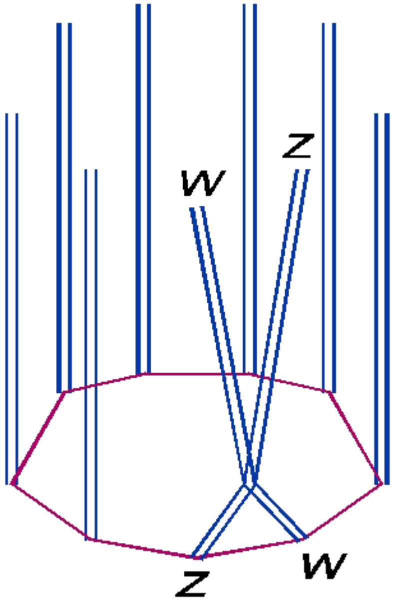


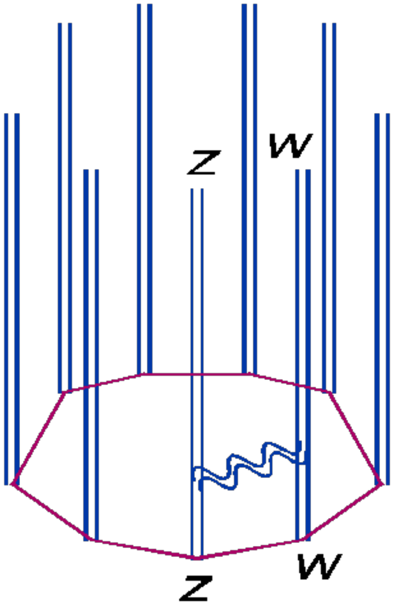
$$\mathrm{tr}\, Z Z Z Z W W Z Z Z W W Z Z Z W W W Z Z W W$$

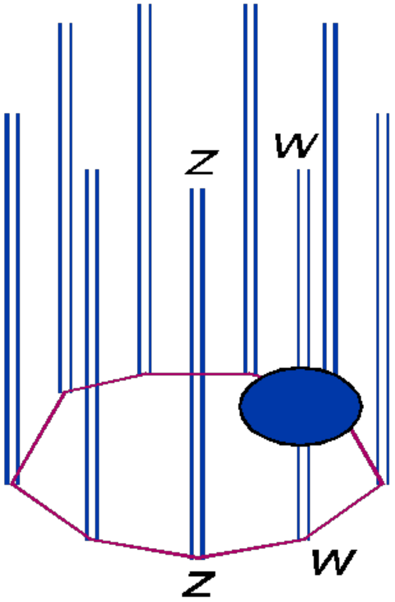


Tree level: $\Delta = L$ (huge degeneracy)

One loop:







One loop planar dilatation generator:

$$\boxed{D = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}(1 - \boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_{l+1}) + O(\lambda^2)}$$

Minahan,Z.'02

Heisenberg Hamiltonian

$$Q = \sum_{l=1}^{L} \boldsymbol{\sigma}_l \cdot [\boldsymbol{\sigma}_{l+1} \times \boldsymbol{\sigma}_{l+2}]$$

Integrability!

$$[D, Q] = 0$$

Bethe equations for Heisenberg model

Rapidity: $$\mathrm{e}^{ip} = \frac{u + i/2}{u - i/2}$$

$$\boxed{\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j \neq k} \frac{u_k - u_j + i}{u_k - u_j - i}}$$

Bethe’31

Anomalous dimension:

$$\Delta - L = \frac{\lambda}{8\pi^2} \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4}$$

Complete spectrum at one loop

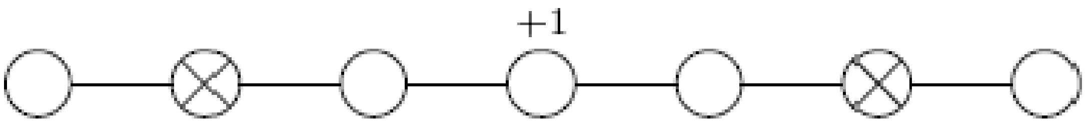
Beisert,Staudacher'03

Nested BAE:

$$\boxed{\left(\frac{u_{\alpha,k}+\frac{iV_\alpha}{2}}{u_{\alpha,k}-\frac{iV_\alpha}{2}}\right)^L=\prod_{(\beta,j)\neq(\alpha,k)}\frac{u_{\alpha,k}-u_{\beta,j}+\frac{iM_{\alpha\beta}}{2}}{u_{\alpha,k}-u_{\beta,j}-\frac{iM_{\alpha\beta}}{2}}}$$

$M_{\alpha\beta}$ - Cartan matrix of PSU(2,2|4)

V_α - highest weight of the field representation



$$\prod_{k,\alpha}\frac{u_{\alpha,k}+\frac{iV_\alpha}{2}}{u_{\alpha,k}-\frac{iV_\alpha}{2}}=1$$

$$\Delta-\Delta_0=\frac{\lambda}{8\pi^2}\sum_{k,\alpha}\frac{V_\alpha}{u_{k,\alpha}^2+V_\alpha^2}$$

Full asymptotic BA

Beisert,Staudacher'05
Beisert,Eden,Staudacher'06
Beisert,Hernandez,Lopez'06

$$1 = \prod_{j\neq k} \frac{s_k - s_j - i}{s_k - s_j + i} \prod_{l} \frac{s_k - r_l + \frac{i}{2}}{s_k - r_l - \frac{i}{2}}$$

$$1 = \prod_{j} \frac{z_k - x_j^+}{z_k - x_j^-} \prod_{l} \frac{r_k - s_l + \frac{i}{2}}{r_k - s_l - \frac{i}{2}}$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k} \mathrm{e}^{\,i\theta(x_k,x_j)} \, \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \, \frac{1 - \frac{1}{x_k^+ x_j^-}}{1 - \frac{1}{x_k^- x_j^+}} \prod_{l} \frac{x_k^- - z_l}{x_k^+ - z_l} \prod_{m} \frac{x_k^- - y_l}{x_k^+ - y_l}$$

$$1 = \prod_{j} \frac{y_k - x_j^+}{y_k - x_j^-} \prod_{l} \frac{v_k - w_l + \frac{i}{2}}{v_k - w_l - \frac{i}{2}}$$

$$1 = \prod_{j\neq k} \frac{w_k - w_j - i}{w_k - w_j + i} \prod_{l} \frac{w_k - v_l + \frac{i}{2}}{w_k - v_l - \frac{i}{2}}$$

$$\theta(x,x') = \sum_{r,s=\pm} rs\,\chi(x_r, x'_s)$$

$$\chi(x,y) = \frac{i}{8\pi^2} \oint_{|z|=1=|w|} \frac{dz}{z}\,\frac{dw}{w}\;\frac{1}{xz-1}\;\frac{1}{yw-1}\;\ln\frac{\Gamma\left(1+\frac{i\sqrt{\lambda}}{4\pi}\left(z+\frac{1}{z}-w-\frac{1}{w}\right)\right)}{\Gamma\left(1-\frac{i\sqrt{\lambda}}{4\pi}\left(z+\frac{1}{z}-w-\frac{1}{w}\right)\right)}$$

$$r = z + \frac{\lambda}{16\pi^2 z} \qquad v = y + \frac{\lambda}{16\pi^2 y} \qquad u = x + \frac{\lambda}{16\pi^2 x} \qquad u \pm \frac{i}{2} = x^\pm + \frac{\lambda}{16\pi^2 x^\pm}$$

$$\Delta - L = \sum_k \left[i(x_k^- - x_k^+) - 1\right] \qquad \prod_k \frac{x_k^+}{x_k^-} = 1$$

Wrapping/finite size effects

$$\Delta_{\text{exact}} = \Delta_{\text{asymptotic BA}} + O(\,\mathrm{e}^{\,-\kappa(\lambda)L})$$

$$\kappa(\lambda) = 2\,\text{arcsinh}\,\frac{\pi}{\sqrt{\lambda}}$$

Ambjørn,Janik,Kristjansen'05

Weak coupling:

$$\Delta = \underbrace{\Delta_0 + \Delta_1\lambda + \ldots +}_{\text{captured by ABA}} O(\lambda^{L})$$

wrapping order

Beisert,Kristjansen,Staudacher'03

Strong coupling:

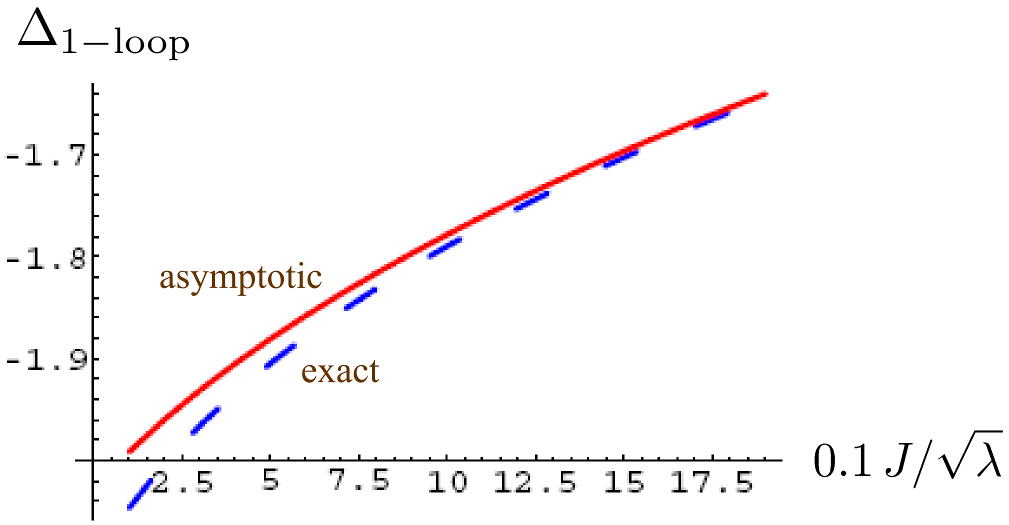
Schäfer-Nameki,Zamaklar,Z.’06

$$\Delta_{\mathrm{exact}} = \Delta_{\mathrm{asymptotic}} + O\left(\mathrm{e}^{-2\pi L/\sqrt{\lambda}}\right)$$

$$\Updownarrow$$

$$\mathrm{e}^{-m\cdot\langle\mathrm{length}\rangle}$$

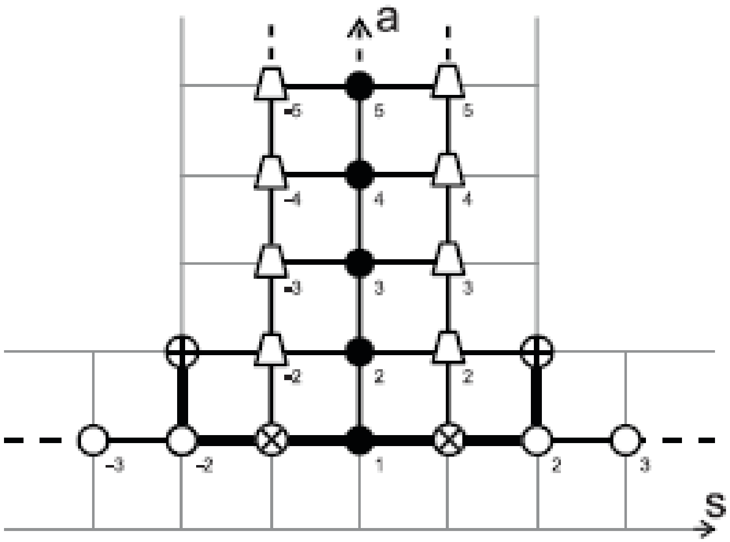
since length of the string $= \dfrac{2\pi L}{\sqrt{\lambda}}$ and mass gap $= 1$ in light-cone gauge



Exact solution

Y-system of thermodynamic Bethe ansatz:

$$\frac{Y^+_{a,s}Y^-_{a,s}}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}\,.$$



Gromov,Kazakov,Vieira’09
Gromov,Kazakov,Kozak,Vieira’09
Bombardelli,Fioravanti,Tateo’09
Arutyunov,Frolov’09

Superconformal Chern-Simons

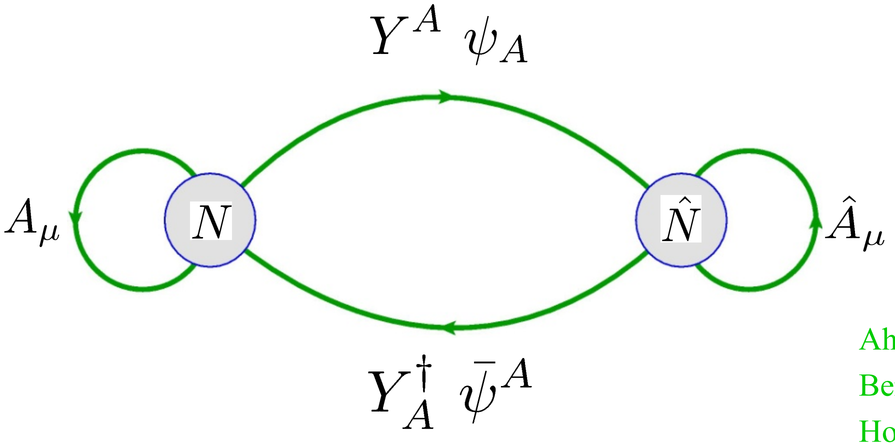
- D=3
- Two gauge groups: $U(N) \times U(\hat{N})$
- Field content:

$$\begin{array}{ccc} A_\mu & & U(N) \\ & \text{in adjoint of} & \\ \hat{A}_\mu & & U(\hat{N}) \end{array}$$

ψ_A, Y^A in bifund. of $U(N) \times U(\hat{N})$

$A = 1 \ldots 4$ spinor index of SO(6) R-symmetry

The Lagrangian



Aharony,Bergman,Jafferis,Maldacena'08;
Benna,Klebanov,Klose,Smedbäck'08;
Hosomichi,Lee,Lee,Lee,Park'08

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \frac{k}{4\pi}\,\mathrm{tr}\,\Bigg[\varepsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}\,A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2}{3}\,\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right) \\
& + D_\mu Y_A^\dagger D^\mu Y^A + i\bar{\psi}^A \not{D}\psi_A + \frac{1}{12}\,Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + \frac{1}{12}\,Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger \\
& - \frac{1}{2}\,Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger + \frac{1}{3}\,Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger \\
& - \frac{1}{2}\,Y_A^\dagger Y^A \bar{\psi}^B \psi_B + Y_A^\dagger Y^B \bar{\psi}^A \psi_B + \frac{1}{2}\,\bar{\psi}^A Y^B Y_B^\dagger \psi_A - \bar{\psi}^A Y^B Y_A^\dagger \psi_B \\
& + \frac{1}{2}\,\varepsilon^{ABCD} Y_A^\dagger \bar{\psi}_{cB} Y_C^\dagger \psi_D - \frac{1}{2}\,\varepsilon_{ABCD} Y^A \bar{\psi}^B Y^C C\psi_c^D\Bigg],
\end{aligned}
$$

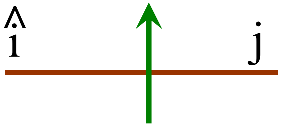
AdS_4/CFT_3 correspondence

Aharony,Bergman,Jafferis,Maldacena'08

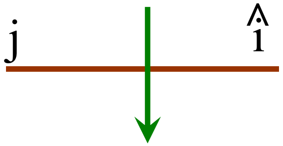
$\mathcal{N}=6$ CS	Strings on $AdS_4 \times CP^3$
't Hooft coupling: $\lambda = \frac{N}{k}$	String tension: $T = \sqrt{\frac{\lambda}{2}}$
Number of colors: N	String coupling: $g_s = \left(32\pi^2\lambda^5\right)^{1/4}\frac{1}{N}$
Difference in ranks: $\hat{N} - N$	World-sheet theta-angle: $\theta = 2\pi(\hat{\lambda} - \lambda)$
Large-N limit	Free strings
Strong coupling	Classical strings
Local operators Scaling dimension: Δ	String states Energy: E

AdS$_4$/CFT$_3$: superconformal Chern-Simons

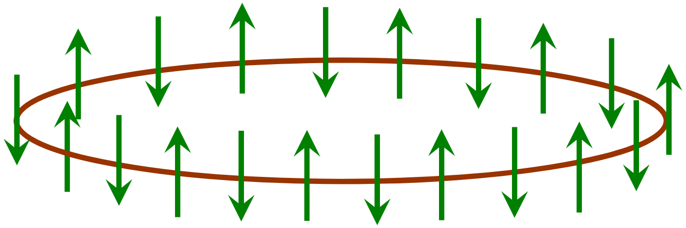
$$Y^{A\,j}_{\hat{i}} \quad \Longleftrightarrow$$



$$Y^{\dagger\,\hat{i}}_{A\,j} \quad \Longleftrightarrow$$



$$\mathcal{O} = \chi^{B_1\ldots B_L}_{A_1\ldots A_L}\ \operatorname{tr} Y^{A_1} Y^{\dagger}_{B_1} \ldots Y^{A_L} Y^{\dagger}_{B_L}$$

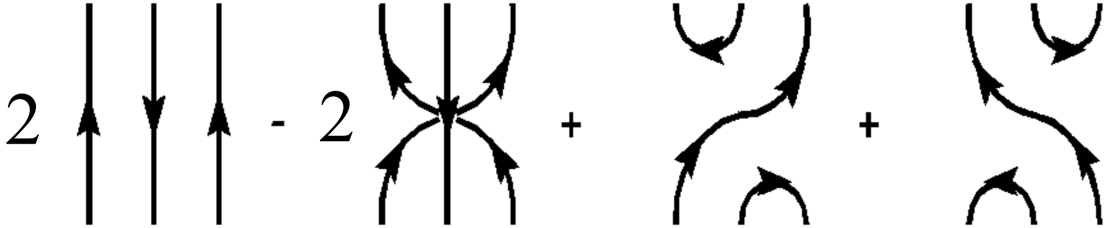


Alternating spin chain of length 2L

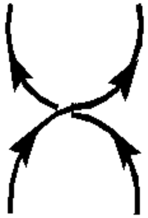
Hamiltonian

$$\boxed{\Gamma = \frac{\lambda\hat{\lambda}}{2}\sum_{l=1}^{2L}\left(2 - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}\right)}$$

Minahan,Z.'08



K = 

P = 

Bethe ansatz equations

$$\left(\frac{u_j+i/2}{u_j-i/2}\right)^L = \prod_{k=1,k\neq j}^{M_u}\frac{u_j-u_k+i}{u_j-u_k-i}\prod_{k=1}^{M_r}\frac{u_j-r_k-i/2}{u_j-r_k+i/2}$$

$$1 = \prod_{k=1,k\neq j}^{M_r}\frac{r_j-r_k+i}{r_j-r_k-i}\prod_{k=1}^{M_u}\frac{r_j-u_k-i/2}{r_j-u_k+i/2}\prod_{k=1}^{M_v}\frac{r_j-v_k-i/2}{r_j-v_k+i/2}$$

$$\left(\frac{v_j+i/2}{v_j-i/2}\right)^L = \prod_{k=1,k\neq j}^{M_v}\frac{v_j-v_k+i}{v_j-v_k-i}\prod_{k=1}^{M_r}\frac{v_j-r_k-i/2}{v_j-r_k+i/2}$$

Kulish,Reshetikhin’83

$$1 = \prod_{j=1}^{M_u}\frac{u_j+i/2}{u_j-i/2}\prod_{j=1}^{M_v}\frac{v_j+i/2}{v_j-i/2}$$

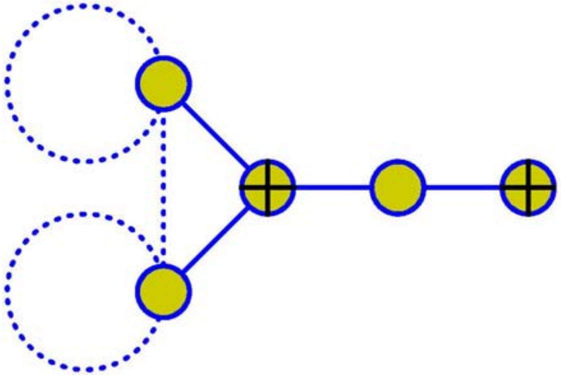
zero-momentum condition

$$\gamma = \lambda\hat{\lambda}\left(\sum_{j=1}^{M_u}\frac{1}{u_j^2+\frac{1}{4}}+\sum_{j=1}^{M_v}\frac{1}{v_j^2+\frac{1}{4}}\right)$$

anomalous dimension

All-loop asymptotic Bethe ansatz

Gromov,Vieira'08



…………… = dressing phase

$$u \to x : \qquad x + \frac{1}{x} = \frac{u}{h(\lambda)}$$

$$u \pm \frac{i}{2} \to x^{\pm} : \qquad x^{\pm} + \frac{1}{x^{\pm}} = \frac{u \pm \frac{i}{2}}{h(\lambda)}$$

$$h(\lambda) = \begin{cases} \lambda + \dots & \lambda \to 0 \\ \sqrt{\frac{\lambda}{2}} + \dots & \lambda \to \infty \end{cases}$$

An unknown interpolating function

$$\begin{aligned}
1 &= \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k} x^+_{4,j}}{1 - 1/x_{1,k} x^-_{4,j}} \prod_{j=1}^{K_{\bar{4}}} \frac{1 - 1/x_{1,k} x^+_{\bar{4},j}}{1 - 1/x_{1,k} x^-_{\bar{4},j}}, \\
1 &= \prod_{j \neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{1,k} - u_{3,j} + \frac{i}{2}}{u_{1,k} - u_{3,j} - \frac{i}{2}}, \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x^+_{4,j}}{x_{3,k} - x^-_{4,j}} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{3,k} - x^+_{\bar{4},j}}{x_{3,k} - x^-_{\bar{4},j}} \\
\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L &= \prod_{j \neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \prod_{j=1}^{K_1} \frac{1 - 1/x^-_{4,k} x_{1,j}}{1 - 1/x^+_{4,k} x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k} - x_{3,j}}{x^+_{4,k} - x_{3,j}} \times \\
&\times \prod_{j=1}^{K_4} \sigma_{\rm BES}(u_{4,k}, u_{4,j}) \prod_{j=1}^{K_{\bar{4}}} \sigma_{\rm BES}(u_{4,k}, u_{\bar{4},j}), \\
\left(\frac{x^+_{\bar{4},k}}{x^-_{\bar{4},k}}\right)^L &= \prod_{j=1}^{K_{\bar{4}}} \frac{u_{\bar{4},k} - u_{\bar{4},j} + i}{u_{\bar{4},k} - u_{\bar{4},j} - i} \prod_{j=1}^{K_1} \frac{1 - 1/x^-_{\bar{4},k} x_{1,j}}{1 - 1/x^+_{\bar{4},k} x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{\bar{4},k} - x_{3,j}}{x^+_{\bar{4},k} - x_{3,j}} \times \\
&\times \prod_{j \neq k}^{K_{\bar{4}}} \sigma_{\rm BES}(u_{\bar{4},k}, u_{\bar{4},j}) \prod_{j=1}^{K_4} \sigma_{\rm BES}(u_{\bar{4},k}, u_{4,j}).
\end{aligned}$$

$$\boxed{\text{for} \quad \lambda = \hat{\lambda}}$$

Why integrability?

D=4

N=4 super-Yang-Mills ⟺ $AdS_5 \times S^5$

D=3

N=6 Super-Chern-Simons ⟺ $AdS_4 \times CP^3$

D=2

$AdS_3 \times S^3 \times T^4$

N=4 symmetric orbifold CFTs ⟺

$AdS_3 \times S^3 \times S^3 \times S^1$

Babichenko,Stefanski,Z.'09

Why these backgrounds?

Semi-symmetric superspaces

Serganova'83

Z_4 symmetric G/H_0 coset:

$$\mathfrak{g} = \overset{\overset{B}{\downarrow}}{\mathfrak{h}_0} \oplus \underset{\underset{F}{\uparrow}}{\mathfrak{h}_1} \oplus \overset{\overset{B}{\downarrow}}{\mathfrak{h}_2} \oplus \underset{\underset{F}{\uparrow}}{\mathfrak{h}_3} \qquad [\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{(i+j)\,\mathrm{mod}\,4}$$

g – coset representative:

$$g^{-1}\partial_\mu g = P_{0\,\mu} + P_{1\,\mu} + P_{2\,\mu} + P_{3\,\mu}.$$

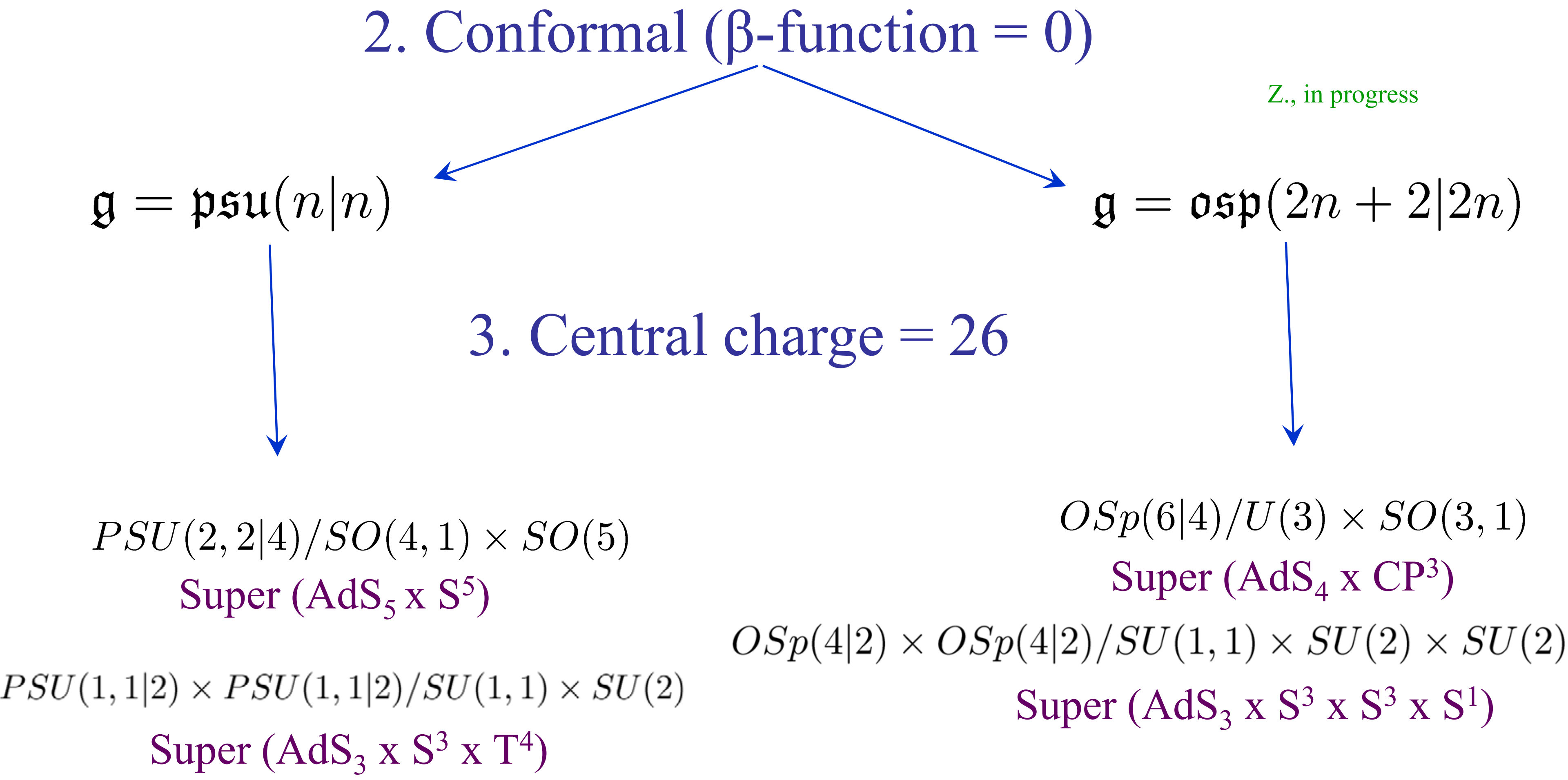
String sigma-model:

$$\mathcal{L} = \frac{1}{2\pi\alpha'}\,\mathrm{Str}\left(\sqrt{-h}h^{\mu\nu}P_{2\,\mu}P_{2\,\nu} + \epsilon^{\mu\nu}P_{1\,\mu}P_{3\,\nu}\right)$$

Metsaev,Tseytlin'98

Berkovits,Bershadsky,Hauer,Zhukov,Zwiebach'99

1. Integrable follows from Z_4 symmetry Bena,Polchinski,Roiban'03



2. Conformal (β-function = 0)

Z., in progress

$\mathfrak{g} = \mathfrak{psu}(n|n)$

$\mathfrak{g} = \mathfrak{osp}(2n+2|2n)$

3. Central charge = 26

$PSU(2,2|4)/SO(4,1) \times SO(5)$

Super (AdS_5 x S^5)

$PSU(1,1|2) \times PSU(1,1|2)/SU(1,1) \times SU(2)$

Super (AdS_3 x S^3 x T^4)

$OSp(6|4)/U(3) \times SO(3,1)$

Super (AdS_4 x CP^3)

$OSp(4|2) \times OSp(4|2)/SU(1,1) \times SU(2) \times SU(2)$

Super (AdS_3 x S^3 x S^3 x S^1)

С
ДНЁМ
РОЖДЕНИЯ!