Bogoliubov compensation principle as a tool for

non-perturbative calculations in gauge theories

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Outlook

1. Introduction.
2. Compensation equation in the EW theory, running EW coupling, applications: muon $g-2$, $\alpha_w(M_W)$.
2. Spontaneous generation of NJL type interaction in QCD.
3. Numerical results.
5. Concluding remarks.

Introduction

N.N. Bogoliubov compensation principle was applied at fist to studies of problems of statistical mechanics. Superfluidity, superconductivity etc.
N.N. Bogoliubov. Soviet Phys.-Uspekhi, 67, 236 (1959); Uspekhi Fiz. Nauk, 67, 549 (1959); Physica Suppl., 26, 1 (1960);
The main principle of the approach is to check if an effective interaction could be generated in a chosen variant of a renormalizable theory. In view of this one performs "add and subtract" procedure for the effective interaction with a form-factor. Then one assumes the presence of the effective interaction in the interaction Lagrangian and the same term with the opposite sign is assigned to the newly defined free Lagrangian.
B.A., Theor.Math.Phys., 140, *1205 (2004); Phys.Atom.Nucl., 69, 1588 (2006); Phys. Lett. B, 656, 67 (2007); Eur.Phys.J.C,* 61, *51 (2009). B.A., M.K.Volkov, I.V.Zaitsev, Int.J.Mod.Phys.A,* 21, *5721 (2006); (ibid)* 24, *2415 (2009).*
N.N. Bogoliubov compensation principle was applied to studies of spontaneous generation of effective non-local interactions in renormalizable gauge theories including QCD and the electro-weak theory.

Compensation equation in the EW theory

We start with EW Lagrangian with $N_{gen} = 3$ lepton and colour quark doublets with gauge group $SU(2)$. That is we restrict the gauge sector to triplet of W^a_μ only.

$$
\begin{aligned}
L = & \sum_{k=1}^{3}\Big(\frac{\imath}{2}\Big(\bar\psi_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar\psi_k\gamma_\mu\psi_k\Big) - m_k\bar\psi_k\psi_k + \frac{g}{2}\,\bar\psi_k\gamma_\mu\tau^a W^a_\mu\psi_k\Big) + \\
& + \sum_{k=1}^{3}\Big(\frac{\imath}{2}\Big(\bar q_k\gamma_\mu\partial_\mu q_k - \partial_\mu\bar q_k\gamma_\mu q_k\Big) - M_k\bar q_k q_k + \frac{g}{2}\,\bar q_k\gamma_\mu\tau^a W^a_\mu q_k\Big) - \\
& - \frac{1}{4}\Big(W^a_{\mu\nu}W^a_{\mu\nu}\Big); \qquad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon_{abc}W^b_\mu W^c_\nu\,. \qquad (1)
\end{aligned}
$$

where we use the standard notations and ψ_k and q_k correspond to left leptons and quarks respectfully. We write here masses for leptons and quarks bearing in mind the ready Higgs phenomenology.

In accordance to the Bogoliubov approach in application to QFT we look for a non-trivial solution of a compensation equation, which is formulated on the basis of the Bogoliubov procedure add – subtract.

$$L = L_0 + L_{int};$$

$$L_0 == \sum_{k=1}^{3}\Big(\frac{\imath}{2}\Big(\bar\psi_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar\psi_k\gamma_\mu\psi_k\Big) - m_k\bar\psi_k\psi_k + \frac{\imath}{2}\Big(\bar q_k\gamma_\mu\partial_\mu q_k -$$

$$\partial_\mu\bar q_k\gamma_\mu q_k\Big) - M_k\bar q_k q_k\Big) - \frac{1}{4}W^a_{\mu\nu}W^a_{\mu\nu} + \frac{G}{3!}\cdot\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}; \qquad (2)$$

$$L_{int} = \frac{g}{2}\sum_{k=1}^{3}\Big(\bar\psi_k\gamma_\mu\tau^a W^a_\mu\psi_k + \bar q_k\gamma_\mu\tau^a W^a_\mu q_k\Big) - \frac{G}{3!}\cdot\epsilon_{abc}W^a_{\mu\nu}W^b_{\nu\rho}W^c_{\rho\mu}. \qquad (3)$$

Here notation $\frac{G}{3!}\cdot\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}$ means corresponding non-local vertex in the momentum space

$$(2\pi)^4 G\,\epsilon_{abc}(g_{\mu\nu}(q_\rho pk - p_\rho qk) + g_{\nu\rho}(k_\mu pq - q_\mu pk) + g_{\rho\mu}(p_\nu qk - k_\nu pq) +$$
$$+q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho)\,F(p,q,k)\,\delta(p+q+k) + ...; \qquad (4)$$

where $F(p,q,k)$ is a form-factor and $p,\mu,a;\ q,\nu,b;\ k,\rho,c$ are respectfully incoming momenta, Lorentz indices and weak isotopic indices of W-bosons.

Effective interaction

$$-\frac{G}{3!}\cdot\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu} \qquad (5)$$

is usually called anomalous three-boson interaction and it is considered for long time on phenomenological grounds. The first attempt to obtain this interaction in the framework of Bogoliubov approach was done in work B.A. Arbuzov, Phys. Lett. B, 288, 179 (1992). There are experimental limitations for G/g.

Let us consider expression (2) as the new free Lagrangian L_0, whereas expression (3) as the new interaction Lagrangian L_{int}. Then compensation conditions will consist in demand of full connected tree-boson vertices of the structure (24), following from Lagrangian L_0, to be zero. This demand gives a non-linear equation for form-factor F.

We use approximation which is similar to that used in the problem of NJL interaction generation.

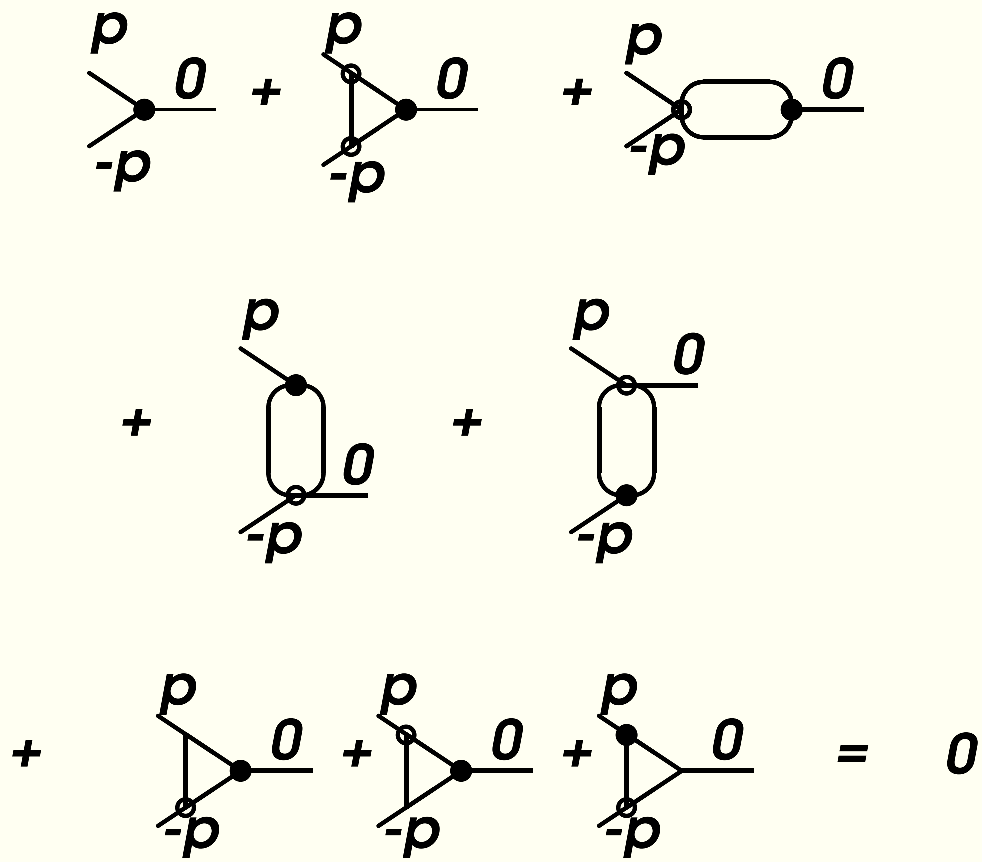

Fig.1. Diagram representation of the compensation equation. Black spot corresponds to anomalous three-boson vertex with a form-factor. Empty circles correspond to point-like anomalous three-boson and four-boson vertices. Simple point corresponds to usual gauge vertex. Incoming momenta are denoted by the corresponding external lines.

Now following the rules being stated above we obtain the following equation for form-factor $F(x)$

$$F(x) = -\frac{G^2 N}{64\pi^2}\Biggl(\int_0^Y F(y)y\,dy - \frac{1}{12x^2}\int_0^x F(y)y^3 dy + \frac{1}{6x}\int_0^x F(y)y^2 dy +$$

$$+\frac{x}{6}\int_x^Y F(y)\,dy - \frac{x^2}{12}\int_x^Y \frac{F(y)}{y}\,dy\Biggr) + \frac{G\,g\,N}{16\,\pi^2}\int_0^Y F(y)\,dy + \tag{6}$$

$$+\frac{G\,g\,N}{24\,\pi^2}\Biggl(\int_{3x/4}^x \frac{(3x-4y)^2(3x-8y)}{x^2(x-2y)}F(y)\,dy + \int_x^Y \frac{(5x-6y)}{(x-2y)}F(y)\,dy\Biggr) +$$

$$+\frac{GgN}{32\pi^2}\Biggl(\int_{3x/4}^x \frac{3(4y-3x)^2(x^2-4xy+2y^2)}{8x^2(2y-x)^2}F(y)dy +$$

$$+\int_x^Y \frac{3(x^2-2y^2)}{8(2y-x)^2}F(y)dy + \int_0^x \frac{5y^2-12xy}{16x^2}F(y)\,dy +$$

$$+\int_x^Y \frac{3x^2-4xy-6y^2}{16y^2}F(y)\,dy\Biggr).$$

Here $x = p^2$ and $y = q^2$, where q is an integration momentum, $N = 2$.

We introduce here an effective cut-off Y, which bounds a "low-momentum" region where our non-perturbative effects act and consider the equation at interval $[0, Y]$ under condition

$$F(Y) = 0. \tag{7}$$

We solve equation (6) by iterations and obtain the following non-trivial solution

$$F(z) = \frac{1}{2} G_{15}^{31}\Big(z\,|^{0}_{1, 1/2, 0, -1/2, -1}\Big) - \frac{85\, g \sqrt{N}}{512\,\pi} G_{15}^{31}\Big(z\,|^{1/2}_{1, 1/2, 1/2, -1/2, -1}\Big) +$$
$$+ C_1\, G_{04}^{10}\Big(z\,|1/2,\, 1,\, -1/2,\, -1\Big) + C_2\, G_{04}^{10}\Big(z\,|1,\, 1/2,\, -1/2,\, -1\Big). \tag{8}$$

$$z = \frac{G^2\, N\, x^2}{1024\,\pi^2}; \quad z_0\, \frac{G^2\, N\, Y^2}{1024\,\pi^2} = 205.42535; \tag{9}$$

$$g(z_0) = -0.43014; \quad C_1 = 0.003687; \quad C_2 = 0.005821. \tag{10}$$

Note that there is also solution with a smaller value of z_0 and large positive $g(z_0)$, which presumably corresponds to strong interaction. This solution is similar to that considered in work PL(2007) and we shall discuss it later on.

We use Schwinger-Dyson equation for W-boson polarization operator to obtain a contribution of additional effective vertex to the running EW coupling constant α_{ew}. The corresponding diagram is presented at Fig.2.

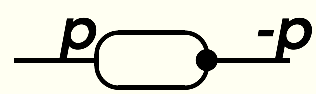

Fig.2. Loop contribution to boson polarization operator. Simple point corresponds to the perturbative vertex

Due to the new vertex being gauge invariant, there is no contribution of ghost fields. So we have after angular integrations

$$\Delta\Pi_{\mu\nu}(x) \;=\; (g_{\mu\nu}\, p^2 - p_\mu p_\nu)\,\Pi(x)\,; \quad x \;=\; p^2\,; \quad y' \;=\; q^2 + \frac{3x}{4}\,;$$

$$\Pi(x) \;=\; -\,\frac{g\,G\,N}{32\,\pi^2}\left(\frac{1}{x^2}\int_{3x/4}^{x}\frac{F(y')\,dy'}{y'-x/2}\left(16\frac{y'^3}{x^2} - 48\frac{y'^2}{x} + 45y - \frac{27}{2}x\right) + \right.$$

$$\left. + \int_x^Y \frac{F(y')\,dy'}{y'-x/2}\left(-\,3y' + \frac{5}{2}\,x\right)\right); \quad g = g(Y)\,. \tag{11}$$

So we have modified one-loop expression for $\alpha_{ew}(p^2)$

$$\alpha_{ew}(x) \;=\; \frac{6\,\pi\,\alpha_{ew}(x_0')}{6\,\pi \,+\, 5\,\alpha_{ew}(x_0')\,\ln(x/x_0') \,+\, 6\,\pi\,\Pi(x)}\,; \quad \alpha_{ew}(x_0) \;=\; \frac{g(Y)^2}{4\,\pi}\,; \tag{12}$$

where x_0' *means a normalization point such that* $\Pi(x_0') = 0$.

Using these expression we calculate behaviour of $\alpha_{ew}(x)$ *down to values of* $x = p^2$ *being by order of magnitude of* M_W.

The new triple vertex contributes to interaction vertex of the Higgs field with

W-s. Considering this contribution we obtain relation of G and M_W

$$G = \frac{\Lambda}{M_W^2}; \quad \Lambda = 0.010312. \tag{13}$$

Substituting (13) into relations (12) we obtain

$$\alpha_{ew}(M_W^2) = 0.0374. \tag{14}$$

It is only 10% larger than well-known value

$$\alpha_{ew}^{exp}(M_W^2) = \frac{\alpha(M_W)}{\sin^2_W} = 0.0337. \tag{15}$$

We consider this result as strong confirmation of the approach. As a matter of fact the accuracy of the present approach was estimated to be just $(10 - 15)\%$.

Now we have new effective interaction of Higgs with W. Due to diagrams of Fig.3 this interaction contributes to muon magnetic moment giving the following additional term in $a = g - 2$

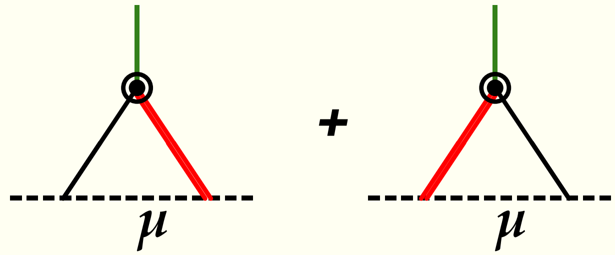

Fig.3. Diagrams for new contribution to muon magnetic moment. Dotted line represents muon (or other spin one-half particle), the green line describes a photon.

$$\Delta a = -\frac{\Lambda\sqrt{2}}{8\,\pi\,\eta}\left(\frac{m_\mu}{M_W}\right)^2 \int_0^{u_0} \frac{\alpha_{ew}(u)\,\Phi(u)\,u\,du}{(u+u_w)(u+u_h)};\; u = \frac{h^2\,p^2}{16\,\pi^2}; \qquad (16)$$

$$u_w = \frac{h^2\,M_W^2}{16\,\pi^2};\; u_h = \frac{h^2\,M_H^2}{16\,\pi^2};\; h = \frac{2\sqrt{2}GgM_W}{\eta};\; \eta = -0.0773;$$

where M_H is yet unknown mass of the Higgs particle. Behaviour (12) of $\alpha_{ew}(Q)$ is presented at Fig.4.

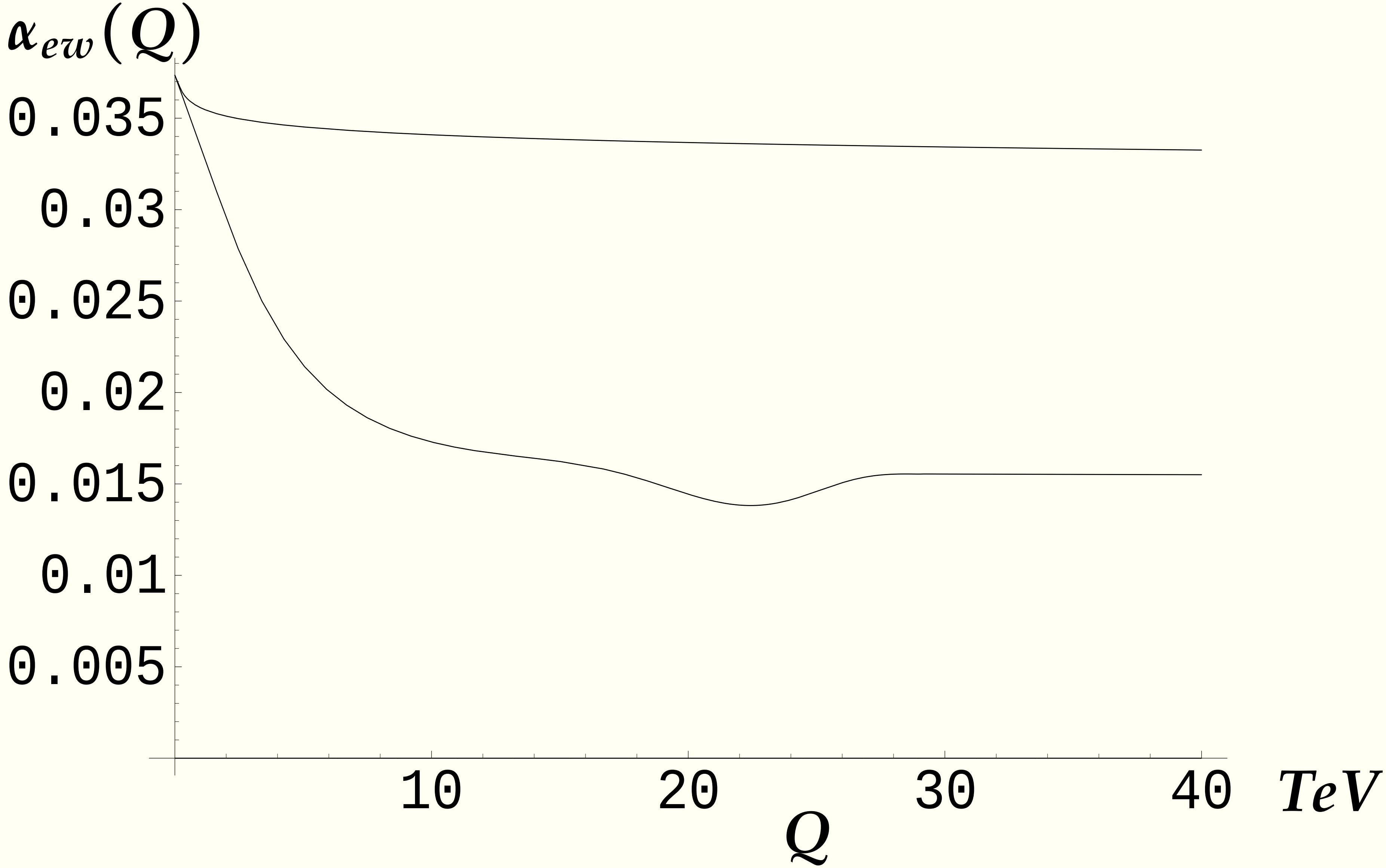

Fig.4. Behaviour of modified $\alpha_{ew}(Q)$ for $0 < Q < 40\,TeV$ with $\alpha_{ew}(M_W) = 0.0374$. The upper line corresponds to usual electroweak coupling with the same normalization.

We know everything but M_H in expression (16) and e.g. for mass of Higgs $M_H = 114\,GeV$ we obtain

$$\Delta a = 3.34 \cdot 10^{-9}; \tag{17}$$

that comfortably fits into error bars for well-known deviation
G.W. Bennett et al., Phys. Rev. D 73, 072003 (2006).
F. Jegerlehner, Acta Phys. Polon. B 38, 3021 (2007).
M. Passera, W.J. Marciano, A. Sirlin, AIP Conf.Proc.1078:378 (2009); arXiv: 0809.4062 (hep-ph) (2008).

$$\Delta a = (3.02 \pm 0.88) \cdot 10^{-9}. \tag{18}$$

With M_H growing Δa (16) slowly decreases inside the error bars down to $2.67 \cdot 10^{-9}$ for $M_H = 300\,GeV$.

Contribution (16) to electron $g-2$ is four orders of magnitude smaller and so it is far below experimental accuracy $\pm 4 \cdot 10^{-12}$.

The same method is applied to the study of non-trivial solution for three-gluon gauge-invariant interaction in QCD. As a result we come to the infrared behavior of $\alpha_s(Q)$ presented at Fig 7. Values for experimental $\alpha_s(Q)$ are extracted from light meson spectrum. Black points: M. Baldicchi, A.V. Nesterenko, G.M. Prosperi, D.V. Shirkov, C. Simolo, PRL, 99, 242001 (2007); the red one: D. Ebert, R.N. Faustov and V.O. Galkin, EPJC, 47, 745 (2006).
Following the same way as previously and using Shirkov-Solovtsov tool for eliminating the well-known perturbative pole we have the following one-loop expression (Fig.5)

$$\alpha_s^{(1)}(x) = \frac{4\pi}{9}\left(\frac{1}{\ln\frac{x}{\Lambda^2}} - \frac{\Lambda^2}{x-\Lambda^2}\right)\left(1+\frac{2g\sqrt{3}}{\alpha_s(x_0')}\left(\frac{1}{\ln\frac{x}{\Lambda^2}} - \frac{\Lambda^2}{x-\Lambda^2}\right)\Pi(x)\right)^{-1}; \tag{19}$$

To estimate contributions of higher loops we take two loop $N_f = 3$ expression and take into account $\Pi(x)$ and Shirkov-Solovtsov tool again, the result is presented at Fig. 6.

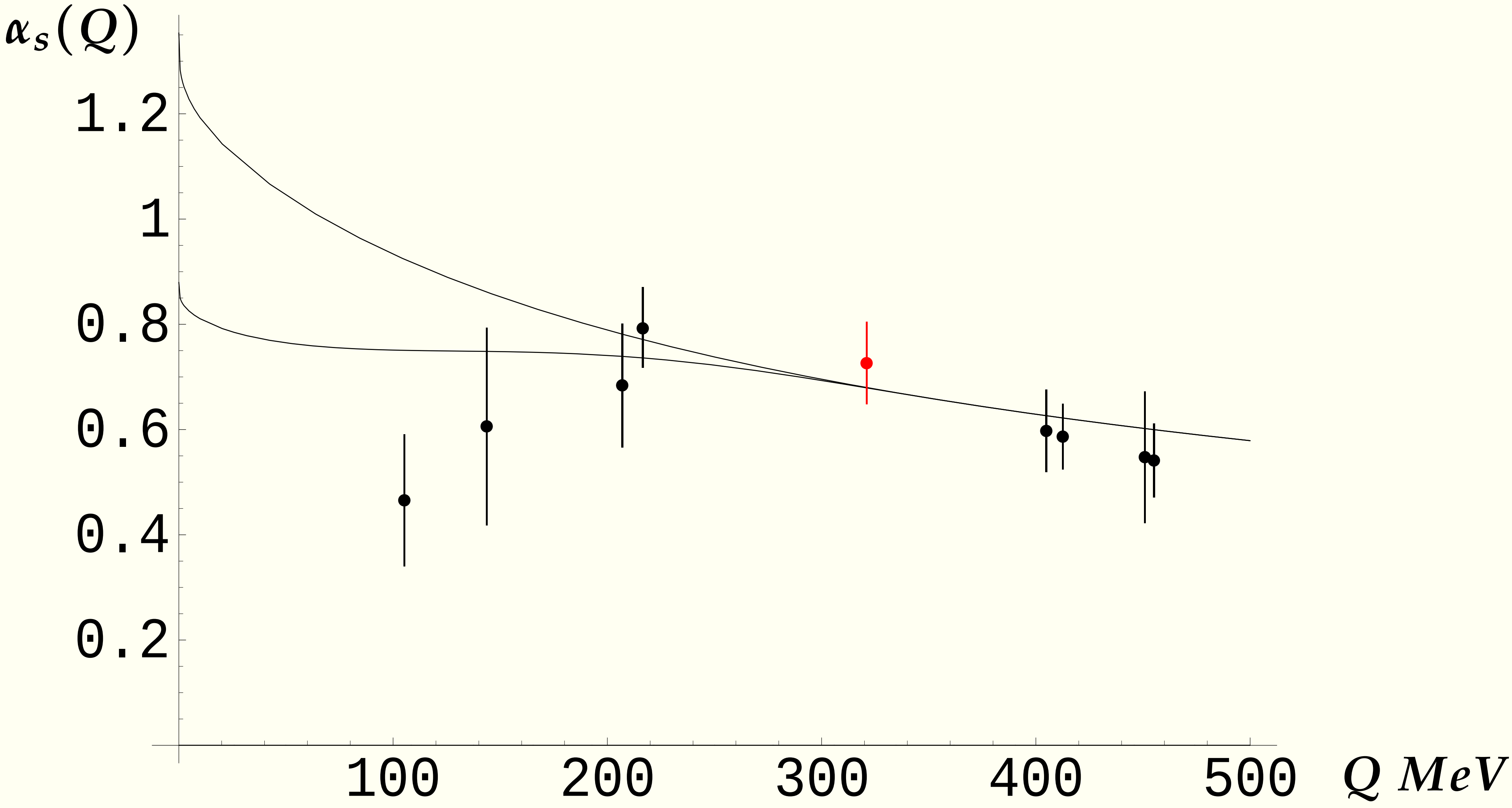

Fig.5. Behaviour of strong coupling $\alpha_s(Q)$, normalised by $\alpha_s(M_\tau) = 0.32$, $0 < Q < 500\,MeV$. Upper line – Shirkov-Solovtsov $\alpha_s(Q)$ with the same normalization. Low momenta freezing $\alpha_s \simeq 0.75$, i.e $\Lambda = 234.7\,MeV$.

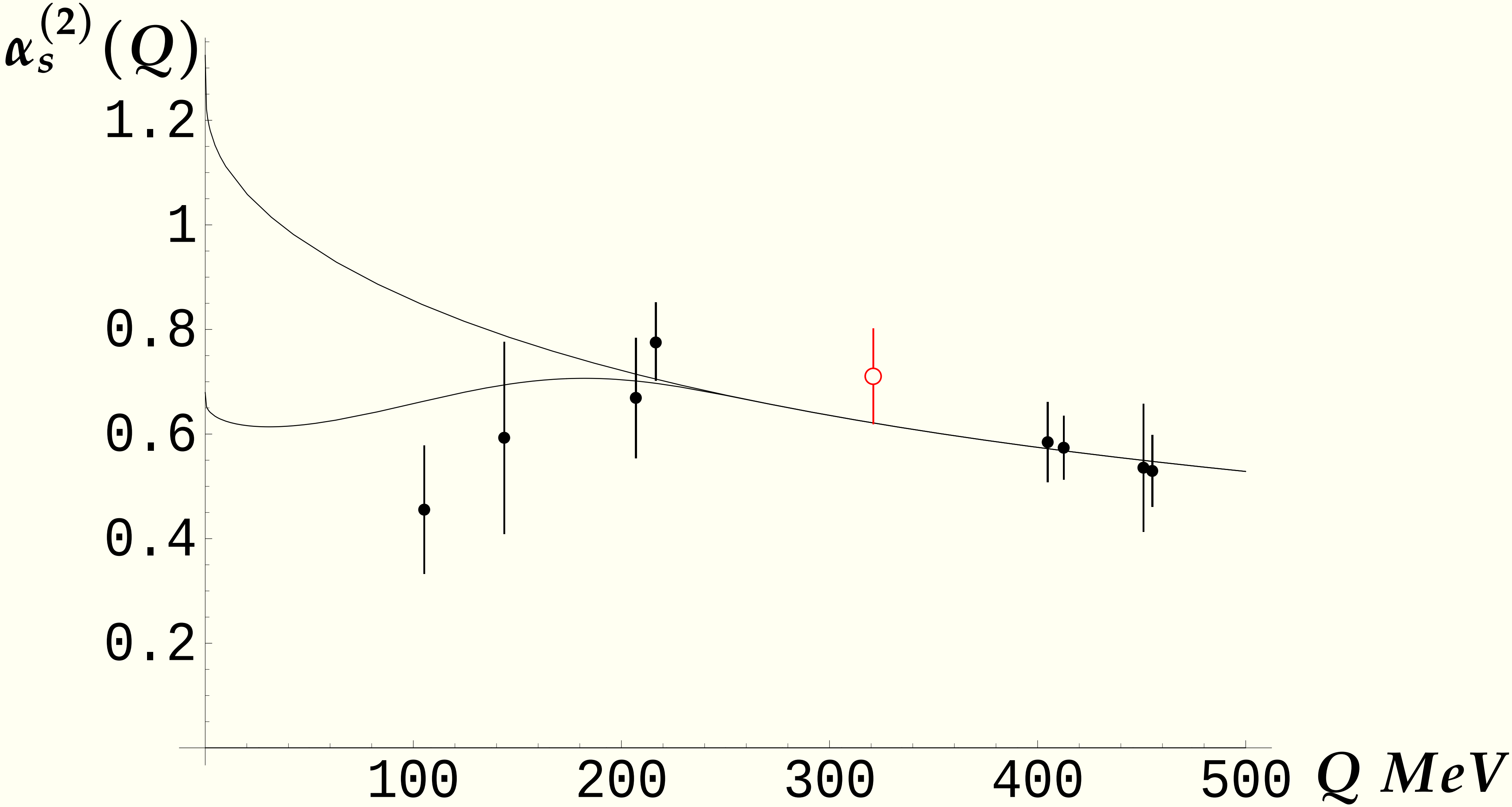

Fig. 6. Model two-loop strong coupling. Notations are the same as at Fig. 5. Upper line model two-loop Shirkov-Solovtsov coupling.

Spontaneous generation of NJL type interaction

We start with QCD Lagrangian with $N_f = 2$ colour quarks with gauge group $SU(3)$.

$$L = \sum_{k=1}^{2}\left(\frac{\imath}{2}\left(\bar{\psi}_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar{\psi}_k\gamma_\mu\psi_k\right) - m_0\bar{\psi}_k\psi_k + \right.$$

$$\left. + g\bar{\psi}_k\gamma_\mu t^a A_\mu^a\psi_k\right) - \frac{1}{4}\left(F_{\mu\nu}^a F_{\mu\nu}^a\right); \qquad (20)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, f_{abc}A_\mu^b A_\nu^c\,.$$

where we use the standard notations and ψ_k correspond to $(u,\, d)$ quarks.

As the next step we apply N.N. Bogoliubov compensation principle to studies of spontaneous generation of effective non-local Nambu – Jona-Lasinio interaction. NJL model proves to be effective in description of low-energy hadron physics. The model starts with effective chiral invariant Lagrangian

$$\frac{G_1}{2}\left(\bar{\psi}\tau^b\gamma_5\psi\,\bar{\psi}\tau^b\gamma_5\psi - \bar{\psi}\,\psi\,\bar{\psi}\,\psi\right); \qquad (21)$$

where ψ is light quark dublet (u, d). This interaction is non-renormalizable, so one is forced to introduce ultraviolet cut-off Λ. Thus we have at least two arbitrary parameters

$$G_1 ; \quad \Lambda_1 ;$$

to be adjusted by comparison with real physics. It comes out that after such adjustment (and similar procedure for vector sector and for s-quark terms) we obtain satisfactory description of light mesons and their low-energy interactions.

However, the problem how to calculate parameters G_i , Λ_i from the fundamental QCD was not solved for a long time. The main problem here is to find a method to obtain effective interactions from fundamental QCD.

There are also non-local variants of NJL model, in which one introduces a form-factor $F(q_i)$ into effective interaction of (1) type instead of a cut-off Λ. In this case again there was no regular method to obtain this function F and one has to do arbitrary assumption for the choice.

Let us assume that a non-local NJL interaction is spontaneously generated in the theory. We use Bogoliubov procedure add – subtract to check this.

$$
\begin{aligned}
L \;&=\; L_0 \,+\, L_{int}\,;\\
L_0 \;&=\; \frac{\imath}{2}\Big(\bar\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\mu\psi\Big) - \frac{1}{4}\,F^a_{0\,\mu\nu}F^a_{0\,\mu\nu} - m_0\bar\psi\,\psi \,+\\
&+ \frac{G_1}{2}\cdot\Big(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi \,-\, \bar\psi\,\psi\,\bar\psi\,\psi\Big) \,+\\
&+ \frac{G_2}{2}\cdot\Big(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\Big)\,; \qquad (22)\\
L_{int} \;&=\; g_s\,\bar\psi\gamma_\mu t^a A^a_\mu\psi \,-\, \frac{1}{4}\Big(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}\Big) \,-\\
&- \frac{G_1}{2}\cdot\Big(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\Big) \,-\\
&- \frac{G_2}{2}\cdot\Big(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\Big)\,. \qquad (23)
\end{aligned}
$$

Here notation** e.g. $\frac{G_1}{2} \cdot \bar\psi\,\psi\,\bar\psi\,\psi$ **means corresponding non-local vertex in the momentum space

$$\imath(2\pi)^4 G_1\,\bar u^a(p)\,u_a(q)\,\bar u^b(k)\,u_b(t)\,F(p,q,k,t)\,\delta(p+q+k+t)\;; \qquad (24)$$

where** $F(p,q,k,t)$ **is a form-factor,** p, q, k, t **are respectively incoming momenta and** a, b **are isotopic indices of corresponding quarks.

Let us consider expression (22) as the new **free** ***Lagrangian*** L_0***, whereas expression (23) as the new*** **interaction** ***Lagrangian*** L_{int}***. Compensation equation*** $\to$ ***full connected four-fermion vertices, following from Lagrangian*** L_0***, to be zero***

1) Perturbative trivial solution $G_i = 0$***.***

2) Non-perturbative non-trivial solution.

The first approximation $\rightarrow$ the following assumptions.

1) Loop numbers 0, 1, 2. For one-loop case only a trivial solution exists.
2) Procedure of linearizing over form-factor, which leads to linear integral equations.
3) Intermediate UV cut-off Λ, results not depending on the value of this cut-off.
4) IR cut-off at the lower limit of integration by Euclidean momentum squared q^2 at value m^2.
5) Only the first two terms of the $1/N$ expansion ($N = 3$).
6)We look for a solution with the following simple dependence on all four variables

$$F(p_1, p_2, p_3, p_4) = F(\frac{p_1^2 + p_2^2 + p_3^2 + p_4^2}{2}); \tag{25}$$

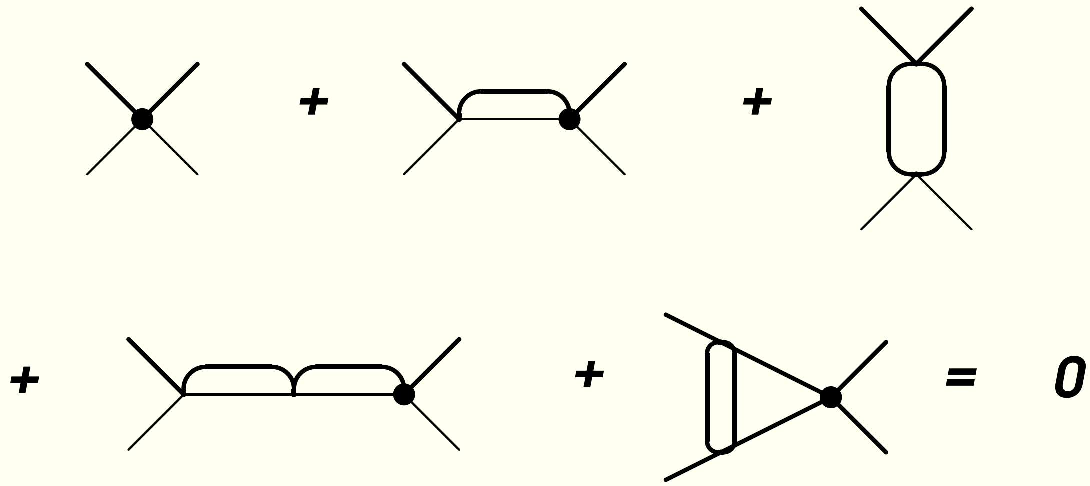

Fig.7. Diagram representation of the compensation equation. Black spot corresponds to anomalous four-fermion vertex with a form-factor. Empty circles correspond to point-like anomalous four-fermion vertices.

After integration in the four-dimensional Euclidean momentum space with IR cut-off at m_0^2. Evident trivial solution $G_1 = 0$.

$$
\begin{aligned}
F_1(x) = {} & A + \frac{3\,G_2}{8\pi^2}\left(2\Lambda^2 + x\log\frac{x}{\Lambda^2} - \frac{3}{2}\,x - \frac{\mu^2}{2x}\right) - \\
& - \frac{(G_1^2 + 6G_1G_2)N}{32\,\pi^4}\left(\frac{1}{6\,x}\int_\mu^x (y^2 - 3\mu^2)F_1(y)\,dy \;+\; \frac{3}{2}\int_\mu^x yF_1(y)\,dy + \right. \\
& + \log\,x\int_\mu^x yF_1(y)\,dy + x\,\log\,x\int_\mu^x F_1(y)\,dy + \int_x^\infty y\log\,y\,F_1(y)\,dy + \\
& + x\int_x^\infty\left(\log\,y + \frac{3}{2}\right)F_1(y)dy + \frac{x^2 - 3\mu^2}{6}\int_x^\infty \frac{F_1(y)}{y}dy + \\
& + \left(2\Lambda^2 - \frac{3}{2}\,x\right)\int_\mu^\infty F_1(y)\,dy - \frac{3}{2}\int_\mu^\infty yF_1(y)\,dy\; - \\
& \left. - \;\log\,\Lambda^2\left(\int_\mu^\infty yF_1(y)\,dy \;+\; x\int_\mu^\infty F_1(y)\,dy\right)\right); \qquad (26)
\end{aligned}
$$

$$
A \;=\; \frac{G_1 N\Lambda^2}{2\pi^2}\left(1 + \frac{1}{4N} - \frac{G_1 N}{2\pi^2}\left(1 + \frac{1}{2N}\right)\int_\mu^\infty F_1(y)\,dy\right);
$$

$$
\mu \;=\; m_0^2; \qquad x \;=\; p^2; \qquad y \;=\; q^2\,.
$$

Existence of a non-trivial solution impose strong

$$u_0 \;=\; 1.92 \cdot 10^{-8} \;\simeq 2 \cdot 10^{-8}\,; \quad G_1 \;=\; \frac{6}{13}\, G_2\,. \qquad (27)$$

We would draw attention to a natural appearance of small quantity u_0. So G_1 and G_2 are both defined in terms of m_0.

Thus we have the unique non-trivial solution of the compensation equation, which contains no additional parameters.

$$F_1(x) = C_1 G_{06}^{40}\left(z\,|1, \frac{1}{2}, \frac{1}{2}, 0, a, b\right) + C_2 G_{06}^{40}\left(z\,|1, \frac{1}{2}, a, b, \frac{1}{2}, 0\right) + \qquad (2$$

$$+C_3 G_{06}^{40}\left(z\,|1, 0, a, b, \frac{1}{2}, \frac{1}{2}\right); \quad a \;=\; -\,\frac{1-\sqrt{1-64u_0}}{4}\,; \quad b \;=\; -\,\frac{1+\sqrt{1-64u}}{4}$$

$$C_1 = 0.283;\; C_2 = -3.65 \times 10^{-8};\; C_3 = -7.79 \times 10^{-8};\; z = \frac{3(G_1^2 + 6 G_1 G_2)x^2}{1024\,\pi^4};$$

Here

$$G_{qp}^{nm}\left(z\,\Big|^{a_1,\dots,a_q}_{b_1,\dots,b_p}\right);$$

is a Meijer function. In case $q = 0$ we write only indices b_i in one line.

It is important, that the solution exists only for positive G_2 and due to (27) for positive G_1 as well. Non-trivial solution $\rightarrow$

$$\begin{aligned} L =\; & \frac{\imath}{2}\Big(\bar\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\mu\psi\Big) - \frac{1}{4}F^a_{0\,\mu\nu}F^a_{0\,\mu\nu} - m_0\bar\psi\,\psi + \\ & + g_s\,\bar\psi\gamma_\mu t^a A^a_\mu\psi - \frac{1}{4}\Big(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}\Big) - \\ & - \frac{G_1}{2}\cdot\Big(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\Big) - \\ & - \frac{G_2}{2}\cdot\Big(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\Big); \end{aligned} \tag{29}$$

$g_s^2/4\,\pi = \alpha_s(q^2)$ ***is the running constant depending on the momentum variable. We need this constant in the low-momenta region. So we use in this region*** $\alpha_s(q^2)$***, which was obtained in the previous section. Possible range of average values of*** α_s 0.50 ***up to*** 0.75.

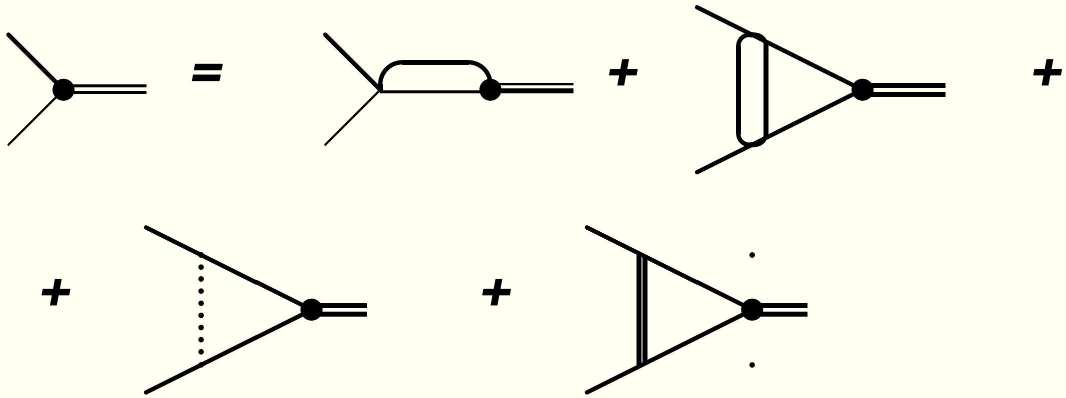

Fig. 8. Diagram representation of Bethe-Salpeter equation for scalar and pseudo-scalar bound state. A meson corresponds to a double line. Dotted line corresponds to the gluon.

Equation for a Bethe-Salpeter wave function $\Psi(x)$ in zero-spin channel →

$$\frac{g^2\,N_c}{4\,\pi^2}\,I_2\;=\;1;\quad I_2\;=\;\int_{m^2}^{\infty}\frac{\Psi(p^2)^2\,dp^2}{p^2}\;=\;\int_{u}^{\infty}\frac{\Psi(z)^2\,dz}{2\,z}\,. \tag{30}$$

$$m_t^2\;=\;-\left(\frac{\alpha_s}{\pi}\;+\;\frac{g^2}{8\,\pi^2}\right)\frac{8\,I_5}{\sqrt{\beta}\,I_2}\,;\quad u=\frac{3(G_1^2+6G_1G_2)m^4}{1024\,\pi^4}\,; \tag{31}$$

where m is a constituent quark mass. I_5 is also a positive integral involving $\Psi(x)$, so we have tachyons with $I = 0, 1$. This means unstable vacuum and after the usual procedure we have expression which connect α_s with integrals involving $\Psi(x)$

$$\alpha_s=\frac{\pi\sqrt{u}}{I_5}\left(1-\left(\frac{u_0}{u}\right)^{\frac{1}{4}}\right)\left(3\left(\frac{u_0}{u}\right)^{\frac{1}{4}}I_3+4\left(1-\left(\frac{u_0}{u}\right)^{\frac{1}{4}}\right)I_4\right)-\frac{\pi}{6I_2}\,. \tag{32}$$

Here all integrals are functions of u and so relation (32) defines function $\alpha_s(u)$.

Provided the non-trivial solution corresponds to the minimal negative value of effective potential while the trivial solution corresponds to its value zero we are to conclude, that just the non-trivial solution is stable and thus the non-trivial solution is to describe the observable physical quantities.

We obtain the expression for pion decay constant f_π.

$$f_\pi = \frac{g\,N}{4\,\pi^2}\int_{m^2}^{\infty}\Big((m-m_0)\,\Psi(y)^2 + m_0\,\Psi(y)\Big)\frac{dy}{y} =$$
$$= \frac{g\,N}{4\,\pi^2}\Big((m-m_0)\,I_2 + m_0\,I_1\Big);\quad I_1 = \int_u^{\infty}\frac{\Psi(z)\,dz}{2\,z}. \tag{33}$$

The pion mass

$$m_\pi^2 = \frac{m^2\,m_0}{2\,\pi\,(m-m_0)\,I_2\,\sqrt{u}}\left(\alpha_s + \frac{g^2}{8\,\pi}\right)I_{\log}; \tag{34}$$

Parameters of the σ-meson

$$m_\sigma^2 = m_\pi^2 + \frac{N\,g^2}{\pi^2}\left(m_0^2\,I_2 + 2\,m_0(m - m_0)\,I_3 + (m - m_0)^2\,I_4\right); \tag{35}$$

$$g_{\sigma\pi\pi} = \frac{g^3\,N}{\pi^2}\left(m_0\,I_3 + (m - m_0)\,I_4\right); \tag{36}$$

$$\Gamma_\sigma = \frac{3\,g^2_{\sigma\pi\pi}}{16\,\pi\,m_\sigma^2}\sqrt{m_\sigma^2 - 4\,m_\pi^2}\,. \tag{37}$$

The quark condensate is also calculated to be

$$< \bar q\,q > = -\left(\alpha_s + \frac{g^2}{8\,\pi}\right)\frac{3\,m^2\,(m - m_0)}{8\,\pi^3\,\sqrt{u}}\,I_6\,; \tag{38}$$

The same approach is applied for vector and axial-vector mesons (AVZ, IJMPA(2009)).

So we have low-energy physical quantities expressed in terms of only two parameters: m_0 and infrared average α_s.

Numerical results

Let us at first discuss results for low-energy QCD. We have expressions for all quantities under study. Then we proceed as follows.
1) We calculate function α_s (32) depending on parameter u and get convinced, that the interesting range of α_s corresponds to u varying in the following region

$$0.0005 \, < \, u \, < \, 0.0015\,. \tag{39}$$

In doing this we use parameter $u_0 = 2\,10^{-8}$ according to relation (27) and define $\Psi(z)$. Having $\Psi(z)$ we calculate integrals I_j, $j = 1, 2, 3, 4, 5$.
2) We fix value $f_\pi = 93\,MeV$.
3) Then for given u in range (39) from (33) we obtain constituent quark mass m.
4) Having m and α_s we calculate m_π from (34).

For u in range (39) m_π varies insignificantly between $134\,MeV$ and $135\,MeV$ with maximal value $134.8\,MeV$ at $u = 0.0009$, that corresponds to $\alpha_s = 0.673$ and $m_0 = 20.27\,MeV$. Considering this value of m_π to be the most suitable, we present a set of calculated parameters for these conditions including quark condensate, parameters of the σ-meson (35), (37) and parameters of ρ and a_1-mesons as well

$$\alpha_s = 0.673\,; \qquad m_0 = 20.3\,MeV\,;$$
$$m_\pi = 135\,MeV\,; \quad m_\sigma = 492\,MeV\,; \quad \Gamma_\sigma = 574\,MeV$$
$$f_\pi = 93\,MeV\,; \quad m = 295\,MeV\,; \quad <\bar{q}\,q>= -\,(222\,MeV)^3\,;$$
$$G_1 = \frac{1}{(244\,MeV)^2}\,; \qquad g = 3.16\,.$$
$$M_\rho = 926.3\,MeV(771.1 \pm 0.9); \quad \Gamma_\rho = 159.5\,MeV(149.2 \pm 0.7);$$
$$M_{a_1} = 1174.8\,MeV(1230 \pm 40); \quad \Gamma_{a_1} = 350\,MeV(250 - 600);$$
$$\Gamma(a_1 \to \sigma\,\pi)/\Gamma_{a_1} = 0.23\,(0.188 \pm 0.043). \tag{40}$$

The upper line here is our input, while all other quantities are calculated (AVZ, IJMPA (2006), (2009)) from these two fundamental parameters. The overall accuracy may be estimated to be of order of 10 – 15%. The worse accuracy occurs in value M_ρ (20%). It seems, that vectors and axials need further study.

We would once more emphasize that all calculated quantities depends on only two initial QCD parameters. The result is obtain exclusively due to application of the Bogoliubov compensation principle and has no analog in any other approach.

To conclude we would like to emphasize that the present approach for the first time permits to determine parameters of effective interaction inherent to the Nambu – Jona-Lasinio model in terms of parameters of the fundamental QCD. The optimal value of $\alpha_s = 0.67$ in (40) agrees with our calculations and it is also quite reasonable from the point of view of the existing knowledge on its low-momenta behaviour. As for value of current quark mass $m_0 \simeq 20\,MeV$, it seems to be rather larger than usual values $m_0(2\,GeV) \simeq 4-8\,MeV$. This problem needs further study.

Let us once more emphasize that we have no additional parameters but really only one dimensional parameter m_0. Thus we derive effective interaction of NJL type from the fundamental QCD and obtain the plausible infrared behavior of α_s.

In addition to QCD results (40) we have also EW results on $\alpha_w(M_W) = 0.0374$ and $\Delta(g-2)_\mu \simeq 3\,10^{-9}$, which also agree with data.

The last results seem even more fundamental than the QCD ones. The calculation of the coupling constant has no analogy in other approaches.

Conclusion

Important result: average value of $\alpha_s \simeq 0.67$ agrees with calculated low-energy α_s. So we have consistent description of low-energy hadron physics with only one dimensional parameter, e.g m_0 or f_π.

Strictly speaking we have also dimensional parameter Λ, which is defined by normalization of α_s at M_τ. However, low-momenta behavior does not depend essentially on this normalization. So for low-momenta NJL model we have really only one parameter (m_0 or f_π).

Basing on the results of the Bogoliubov principle approach we would make two essential conclusions.

Firstly, a subsequent development of the present approach to the hadron physics quite deserves attention. In particular it is advisable to do further applications of the approach to calculation of parameters of vector mesons ρ, ω, A_1 and of the gluon condensate etc.. In the electro-weak theory it is advisable to consider effective NJL-like interaction for heavy quarks (t, b). These problems comprise subjects for forthcoming studies. Secondly, the *positive result of applicability test with Nambu – Jona-Lasinio model and the electroweak theory allows to hope for successful application of the approach to other problems.*

The very final remark. An existence of a non-trivial solution is extremely restrictive. In most cases such solutions do not exist at all. This explains why parameters of the problems under consideration become fixed. When we start from a renormalizable theory we have arbitrary value for its coupling constant. Provided there exists stable non-trvial solution of a compensation equation the coupling is fixed as well as the parameters of this non-trivial solution.

Bearing this in mind we could to dream. Why we could not find in this way a unique stable theory, in which all dimensionless parameters are to be calculated. Then everything is defined by one (or few) dimensional parameter(s). We would insist that such aim might be achieved only by application of the Bogoliubov approach.

Appendix

Now it is the proper place to comment the problem of stability. From the very beginning we have two solutions: the trivial one $G_1 = G_2 = 0$, $m = m_0$ *and the non-trivial one. The constant term in effective potential (??) is connected with the following vacuum averages*

$$C = \frac{1}{4} < F^a_{\mu\nu} F^a_{\mu\nu} >_q + \Sigma_i\, m_i < \bar{\psi}_i\, \psi_i >; \qquad (41)$$

where the first term represent quark induced part of the gluon condensate. To estimate value of effective potential (??) we take numbers (40) for α_s, m *and use expression (41) with the following phenomenological values*

$$g^2 < F^a_{\mu\nu} F^a_{\mu\nu} >_q = 0.3\, GeV^4; \qquad < \bar{q}\, q > \; - (240\, MeV)^3. \qquad (42)$$

The light quarks contribution to the gluon condensate is estimated to be $2/3$ *of its standard value under assumption. Substituting these values into (41) we obtain* $C = 0.1\, m^4$*, while the second part of the potential (??) at the position of minimum gives for values (40)* $V_\xi = -0.16\, m^4$*. So* $V = C + V_\xi = -0.06\, m^4$

and we conclude, that for $\alpha_s = 0.673\,; \quad m_0 = 20.3\,MeV$ the non-trivial solution is stable. It is important to consider the dependence of stability on value of α_s. By direct calculations we get convinced that for $\alpha_s \rightarrow 0$ V_ξ also tends to zero remaining negative. In the present work we obtain only $<\bar{u}\,u>$, $<\bar{d}\,d>$ and modules of these quantities do not decrease with $\alpha_s \rightarrow 0$. Provided the other vacuum expectation values have similar behaviour we may expect, that for some value of $\alpha_s = \alpha_s^{crit}$ the minimum value of effective potential V passes through zero and then becomes positive. This means phase transition to the trivial solution. So in the framework of the present approach the existence of a critical value of α_s is quite natural. For $\alpha_s < \alpha_s^{crit}$ the trivial solution is valid and for $\alpha_s > \alpha_s^{crit}$ just the non-trivial solution with Nambu – Jona-Lasinio interaction is realized.