Quantum surfaces related to the Schrödinger equation

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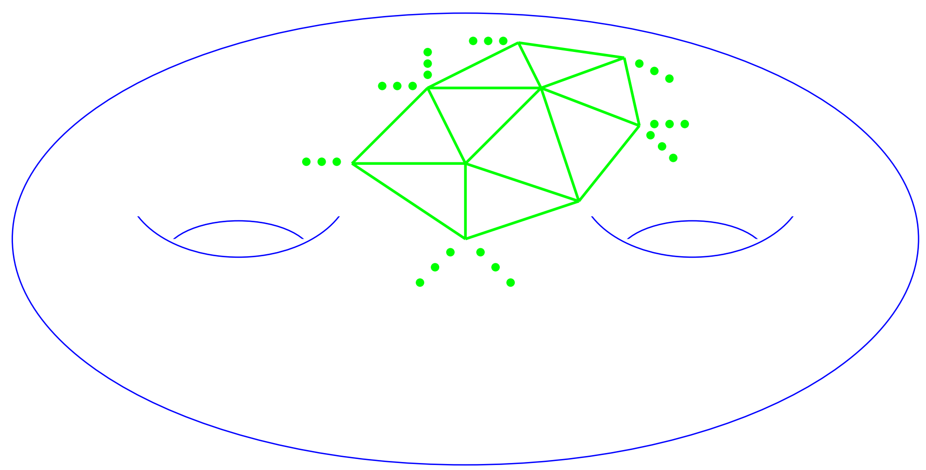
- Matrix models and their applications to problems of physics and geometry
- The β-model and AGT: perturbative approach
- The β-model and Ricatti equation: Quantum surfaces as a nonperturbative approach
- Algebraic geometry of QS; constructing their symplectic invariants $\mathcal{F}_g$

Matrix models is a technique for computing "action functionals" and correlation functions appearing in physics and applications. Loosely speaking, main idea is to replace functionals of 2 variables with matrices with two indices.

The Einstein action in 2D gravity (over all possible metrics and topologies)

$$\int dg e^{-\kappa \int \sqrt{-g} d^2 x} = e^{\mathcal{F}}$$

is approximated by the sum over **triangulations** of surfaces of all genera,



't Hooft idea of $1/N$ expansion. In the Matrix integral

$$\int_{N\times N} DH\, e^{-N\, tr\, V(H)} = e^{\sum_{g=0}^{\infty} N^{2-2g} \mathcal{F}_g} := \mathcal{Z}_{1MM}^{(N)}\big(\{t_k\}\big), \quad W(x) = \sum_{k=1}^{p} \frac{1}{k} t_k x^k$$

contributions of different genus enter with different powers of N. (The order is the Euler characteristic of the corresponding triangulated surface.)

Applications to physical processes:

- Hermitian two-matrix model: Hele–Shaw cell—spreading of water in a viscous liquid (oil) (A. Zabrodin, P. Wigman) [fluid mechanics]
- M-theory (Nekrasov et al); **AGT correspondence 09-10** [string theory]
- Ising model on random surfaces (V.Kazakov, I.Kostov) [statistical mechanics]
- applications to higher-dimensional QCD: (IKKT multi-matrix model) [higher-dimensional quantum field theories]
- Asymptotics of large partitions (Okounkov [Fields medal], Tracy and Widom) [representation theory]

Mathematical structures encoded in $\mathcal{F}$ are rich:

—classical orthogonal polynomials: in terms of N-fold integration over eigenvalues x_i of H we have

$$\mathcal{Z}_{1MM}^{(N)}\bigl(\{t_k\}\bigr) = \int dx_1 \dots dx_N \, \Delta(x)^2 \, e^{N\sum_{i=1}^{N} V(x_i)} = \prod_{i=0}^{N-1} h_i$$

—Integrable systems: Partition functions of matrix models are tau-functions of integrable hierarchies

—Geometry: Kontsevich matrix model (with an external matrix field)

$$\mathcal{Z}_{\mathsf{K}}(\{T_k\}) = \int DH \, e^{-N\,tr\left(\frac{1}{2}\Lambda H^2 + \frac{i}{3}H^3\right)}, \qquad T_k = \frac{1}{2k+1}\, tr\,\Lambda^{-2k-1},\ k = 1, 2, \dots .$$

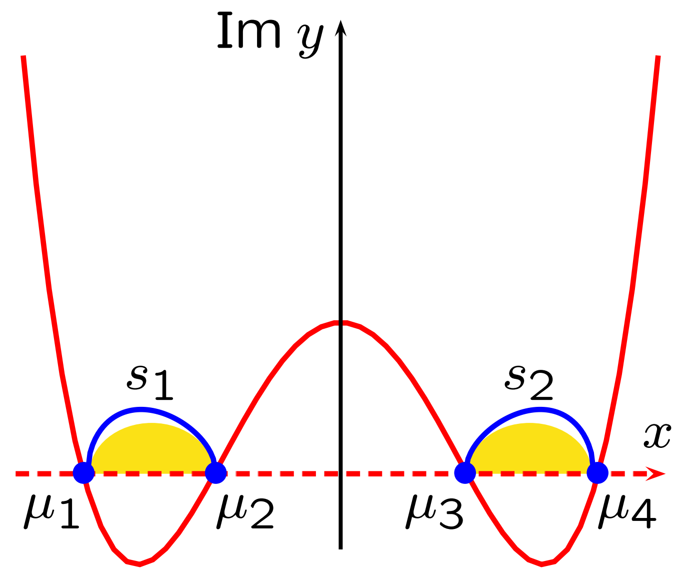
is central in Kontsevich's proof [Fields medal] of the Witten hypothesis that the generating function for intersection indices on moduli spaces (= correlation functions for 2D gravity) satisfies the KdV hierarchy equations.

L.Ch.+Yu.Makeenko'92. The model

$$\int DH \, e^{-\alpha N\,tr\left(\frac{1}{2}\Lambda H \Lambda H + \log(1-H) + H\right)}$$

generates intersection indices for discretized moduli spaces and admits the Givental decomposition [L.Ch.'95]

The asymptotic distribution of eigenvalues $\rho(x) = \mathrm{Im}\, y(x)$ spans in general n intervals (multigap solutions) providing the spectral curve—a hyperelliptic Riemann surface $y^2 = \prod_{j=1}^{2n}(x - \mu_j)$



– $\mathcal{F}_0$ – satisfies equations of the Whitham–Krichever hierarchy w.r.t. s_i—the occupation numbers—and t_k (the times of the potential) [L.Ch., A.Mironov]'02

– $\mathcal{F}_0$ – satisfies Witten–Dijkgraaf–Verlinde–Verlinde equations w.r.t. s_i and t_k (the times of the potential) [L.Ch., A.Marshakov, A.Mironov, D.Vassiliev]'03

– $\mathcal{F}_0$ for matrix theories with hard edges also satisfy WDVV equations w.r.t. s_i and t_k (the times of the potential) [L.Ch.]'05

Asymptotic $(N \to \infty)$ methods for solving matrix integrals

Solving the *loop equation* [Makeenko, Migdal'77] (= infinite set of Virasoro constraints on $Z^{(N)}_{1MM}$) in terms of moments: [Ambjørn, L.Ch., Yu.Makeenko, Ch.Kristjansen]: *all* $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, etc. are expressed via $M^{(j)}_k := y^{(k)}(\mu_j)$, $k = 1, 2, \ldots$.

- $\mathcal{F}_0$ in the multicut case

$$\mathcal{F}_0 = \frac{1}{2}\oint_{\mathcal{C}_D} y(\xi)W(\xi) = \sum_{i=1}^{n} \frac{1}{2} s_i^2 \log s_i + \text{pert.}$$

- $\mathcal{F}_1$ in the multicut case [L.Ch.]'03

$$\mathcal{F}_1 = -\frac{1}{24}\log\left(\prod_{j=1}^{2n} M^{(j)}_1 \Delta(\mu)^4 (\det_{i,j=1,\ldots,n-1} \sigma_{j,i})^{12}\right), \qquad \sigma_{i,k} \equiv \oint_{A_i} \frac{\xi^{k-1}}{y(\xi)} d\xi.$$

- new general procedure of finding $\mathcal{F}_g$ using Feynman-like diagrams [B. Eynard, L.Ch.]'05

We define the **one-point resolvent** to be a 1-*differential*

$$W_1(\lambda) = \hbar \left\langle \sum_{i=1}^{N} \frac{1}{\lambda - x_i} \right\rangle d\lambda, \quad \hbar = t_0/N,$$

and the s-point resolvents ($s \geq 2$) to be symmetric s-differentials

$$W_s(\lambda_1, \dots, \lambda_s) = \hbar^{2-s} \left\langle tr \frac{1}{\lambda_1 - H} \cdots tr \frac{1}{\lambda_s - H} \right\rangle_{\text{conn}} d\lambda_1 \cdots d\lambda_s$$

(“conn"means the connected part of a correlation function). All the W's have the genus expansions $W_s(\lambda_1, \dots, \lambda_s) = \sum_{h=0}^{\infty} \hbar^{2h} W_s^{(h)}(\lambda_1, \dots, \lambda_s)$

- **Loop equation** expresses invariance under the change of integration variables $\delta x_i = \epsilon \frac{1}{x_i - x}$ and is exact:

$$W_1^2(x) - V'(x) W_1(x) + \left\langle tr \frac{V'(x) - V'(H)}{x - H} \right\rangle + \hbar^2 W_2(x, x) = 0.$$

For $W_1^{(0)}(x) = y(x) + V'(x)/2$ we obtain **algebraic equation** determining the **spectral curve**:

$$y^2(x) = \frac{1}{4} V'(x)^2 + P_{n-1}(x) \equiv U(x)$$

The variables t_k, s_i are: $t_k = \text{res}_\infty x^{-k} y(x)$, $k \geq 0$; $s_i = \oint_{A_i} y(x) dx$.

Iterative solution of the loop equation:

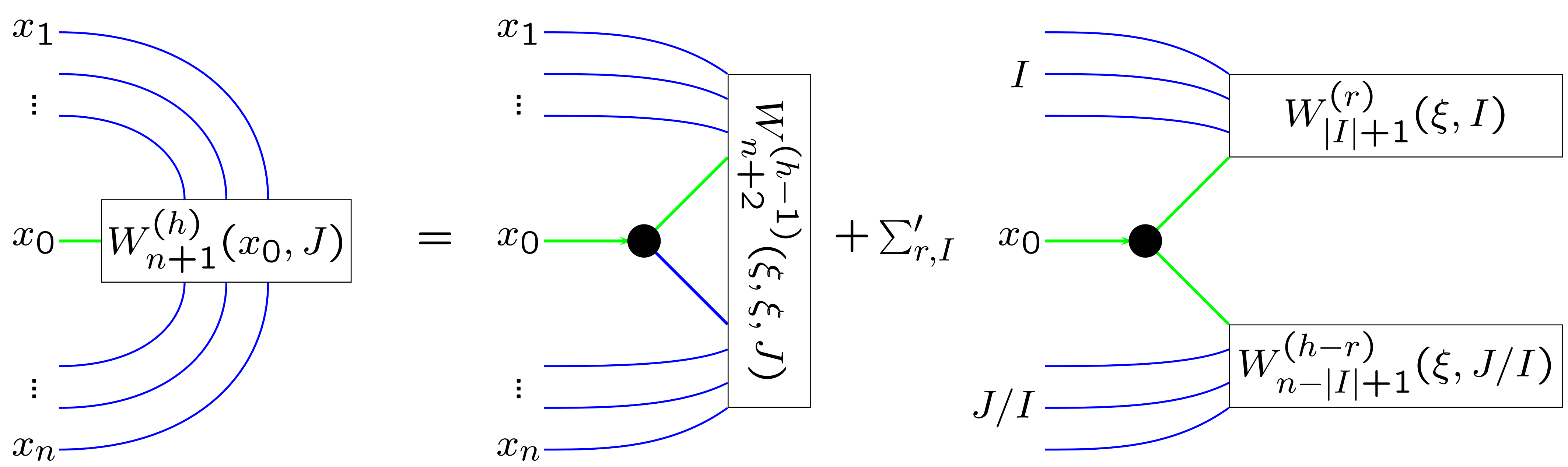
$$W_2^{(0)}(x,y) = B(x,\bar{y}),$$

for P and Q point on the spectral curve, $B(P,Q)$ is the Bergmann bi-differential symmetric in $P \leftrightarrow Q$, canonically normalized, $\oint_{A_i} B(\cdot,Q) = 0$, and such that

$$B(P,Q)\Big|_{P\to Q} = \left(\frac{1}{(\xi(P)-\xi(Q))^2} + O(1)\right) d\xi(P)d\xi(Q),$$

with no other singularities. $\bar{y}$ denotes the point on the second sheet of the hyper-elliptic curve.

Solution to the loop equation reads (in the graphic form)



Symplectic invariants $\mathcal{F}_h$

The above recurrent relation on the resolvents reads

$$W^{(h)}_{n+1}(x_0,J) = \sum_i \mathop{res}_{\xi\to\mu_i} \frac{dE_{\xi,\bar\xi}(x_0)}{2(y(\xi)-y(\bar\xi))d\xi}\left[W^{(h-1)}_{n+2}(\xi,\bar\xi,J) + {\sum_{s,I\subseteq J}}' W^{(s)}_{|I|+1}(\xi,I)W^{(h-s)}_{n-|I|+1}(\bar\xi,J/I)\right],$$

the sum ranges only *stable* correlation functions $W^{(a)}_b$ with $a\geq 0$, $b>0$, and $2a+b-2>0$.

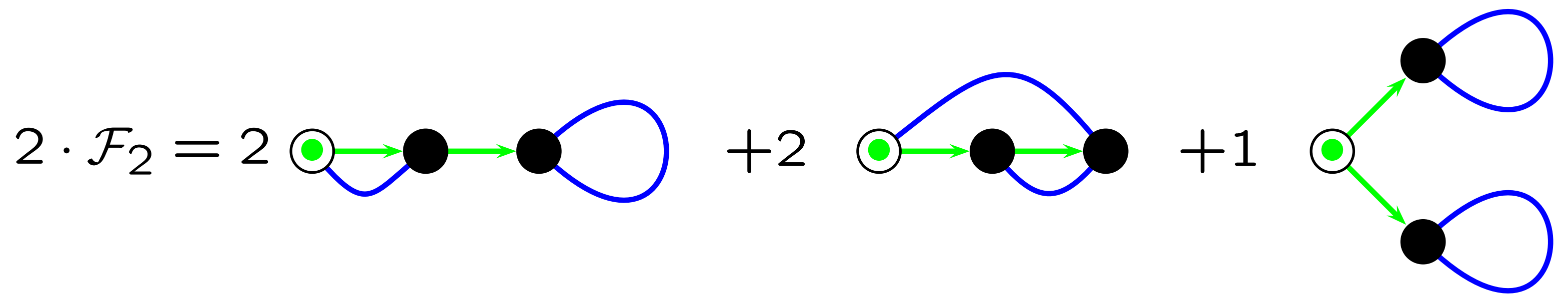
We use the new operator $H\cdot$ for **inverting** the loop insertion operator:

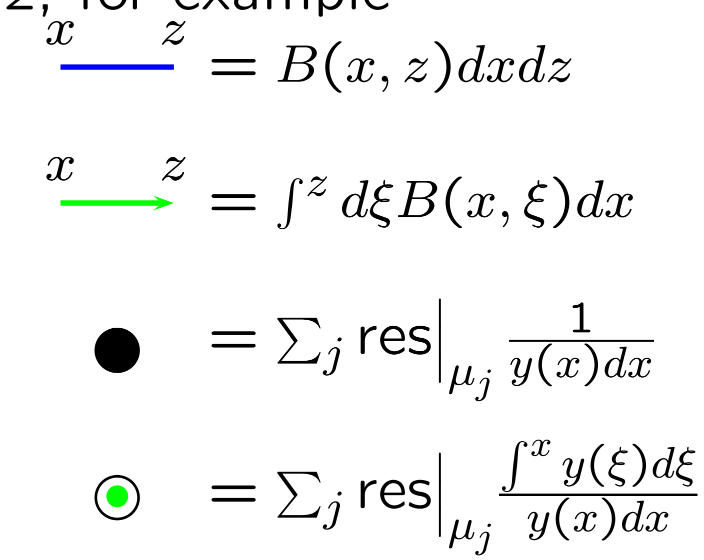
$$H\cdot\varphi := \mathop{res}_{\infty_x} V(x)\varphi(x) - \mathop{res}_{\infty_{\bar x}} V(x)\varphi(x) + t_0\int_{\infty_x}^{\infty_{\bar x}}\varphi(x)dx + \sum_{i=1}^{g} s_i\oint_{B_i}\varphi(x)dx,$$

then

$$\mathcal{F}_h = \frac{1}{2h-2}H_x\cdot W^{(h)}_1(x).$$

We then have the diagrammatic expression for $\mathcal{F}_h$ with $h \geq 2$; for example





[green arrows indicate the order of taking residues]

Here $B(x,z)dx\,dz$ is the normalized bi-differential on the Riemann surface Σ with the double pole as $x \to z$, $y(x)dx$ is the 1-differential on Σ, which is a **hyperelliptic** Riemann surface in the one-matrix model case.

List of identities:

- $\frac{\partial y(x)dx}{\partial s_i} = dw_i(x)$; $dw_i(x)$ are canonically normalized 1-differentials, $\oint_{A_i} dw_j = \delta_{i,j}$, $\oint_{B_i} dw_j = \tau_{i,j}$ – the (symmetric) period matrix;

- $$\frac{\partial W_t^{(h)}(I)}{\partial s_i} = \frac{1}{2\pi i}\oint_{B_i} W_{t+1}^{(h)}(\xi, I) \text{ for any } h, t.$$

  In particular,

  $$\frac{\partial \mathcal{F}_h}{\partial s_i} = \frac{1}{2\pi i}\oint_{B_i} W_1^{(h)}(\xi);$$

- $\oint_{B_i} B(P,Q) = 2\pi i\, dw_i(P)$.

Same technique works for

—finding $\mathcal{F}_h$ in one-matrix model with hard edges [L.Ch.]'05

—finding $\mathcal{F}_h$ in the two-matrix model (here Σ is an arbitrary algebraic curve) [B.Eynard, L.Ch., N.Orantin]'06

$$\int DH_1\, DH_2\, e^{-N\, tr(V_1(H_1)+V_2(H_2)+H_1H_2)}$$

- general procedure of finding $\mathcal{F}_h$ in the β eigenvalue model using Feynman-like diagrams [B.Eynard, L.Ch.]'06

$$\int_N dx_i\, |\Delta(x)|^\beta\, e^{-\frac{N\sqrt{\beta}}{t_0}\sum_{i=1}^N V(x_i)} \qquad \beta = \begin{cases} 1 - \text{orthogonal matrices} \\ 2 - \text{Hermitian matrices} \\ 4 - \text{symplectic matrices} \end{cases},$$

for arbitrary β **and any potential for which V' is a rational function** [this includes the AGT-conjecture case], we know the answer for $\mathcal{F}_{g,k}$, where $\mathcal{F} = \sum_{g,k=0}^{\infty} N^{2-2g-k}(\sqrt{\beta} - \sqrt{\beta}^{-1})^k \mathcal{F}_{g,k}$.

Quantum surfaces = nonperturbative solutions of the β-eigenvalue model[L.Ch.,B.Eynard,O.Marchal]'09-10

- **Loop equation** expresses invariance under the change of integration variables $\delta x_i = \epsilon \frac{1}{x_i - x}$ and is exact:

$$W_1^2(x) - V'(x)W_1(x) + \left\langle tr \frac{V'(x) - V'(H)}{x - H} \right\rangle + (\sqrt{\beta} - \sqrt{\beta}^{-1})W_1'(x) + \hbar^2 W_2(x,x) = 0.$$

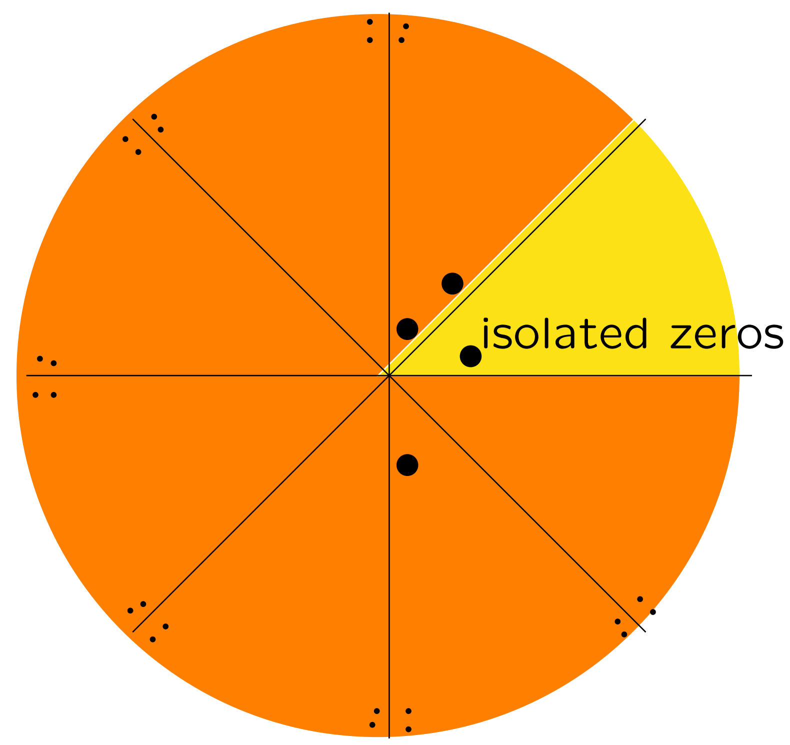
For $W_1^{(0)}(x) = y(x) + V'(x)/2$ we obtain **Ricatti equation** determining the **spectral curve**:

$$y^2(x) + \hbar y'(x) = \frac{1}{4}V'(x)^2 + P_{n-1}(x) \equiv U(x), \text{ where we identify } \hbar = \sqrt{\beta} - \sqrt{\beta}^{-1}.$$

Solution is $y(x) = \psi'(x)/\psi(x)$, where $\psi(x)$ solves the **Schrödinger equation** $\hbar^2\psi''(x) = U(x)\psi(x)$.

- **Stokes Sectors** We choose the function $\psi_\alpha(x)$ to be the solution of the Schrödinger equation that decreases at the αth sector

$$S_k = \left\{ \mathrm{Arg}(x) \in \left] -\frac{\theta_0}{d+1} + \pi\frac{k-\frac{1}{2}}{d+1}, -\frac{\theta_0}{d+1} + \pi\frac{k+\frac{1}{2}}{d+1} \right[\right\}$$



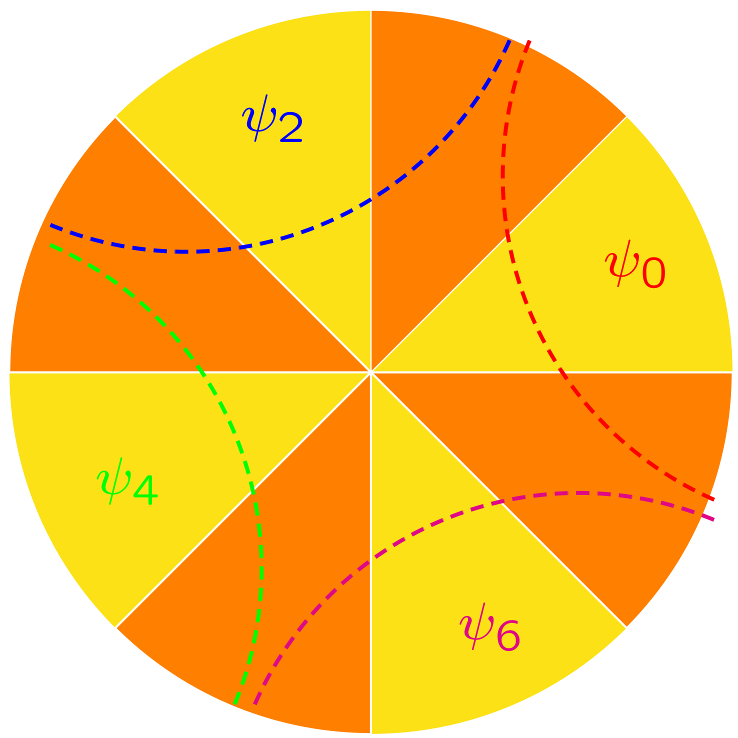
- We define $y(x)$ **sectorwise**:

$$\omega(\overset{\alpha}{x}) := W_1^{(0)}(\overset{\alpha}{x}) = \hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \frac{V'(x)}{2}, \text{ for } x \in S_\alpha.$$

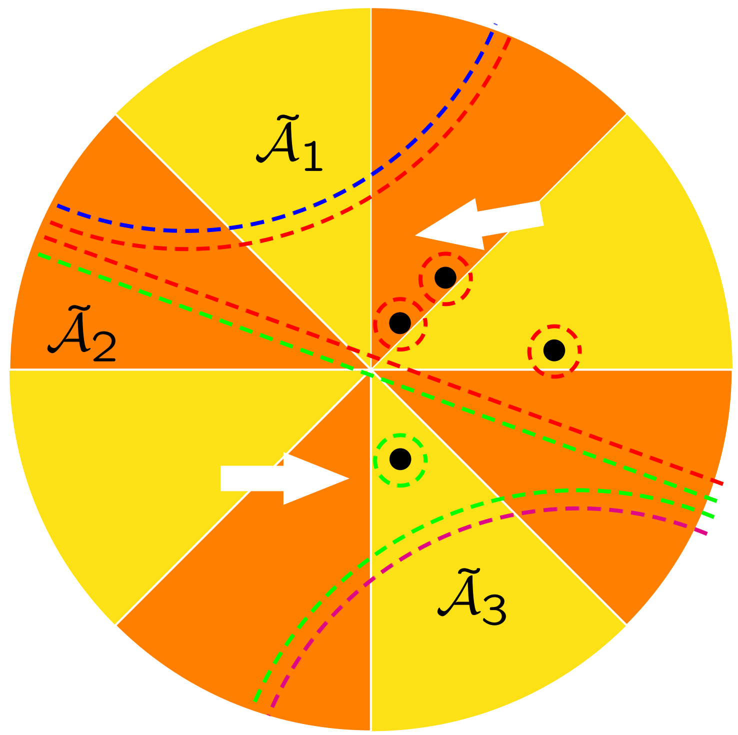
- The contour $\mathcal{C}_D$ and the set of $\mathcal{A}$- and $\mathcal{B}$-cycles

$$\oint_{\mathcal{C}_D} f(x)dx \equiv \sum_{\alpha} \int_{\infty_{\alpha-1}}^{\infty_{\alpha+1}} f(\overset{\alpha}{x})dx$$

The original integration contour $\mathcal{C}_D$:



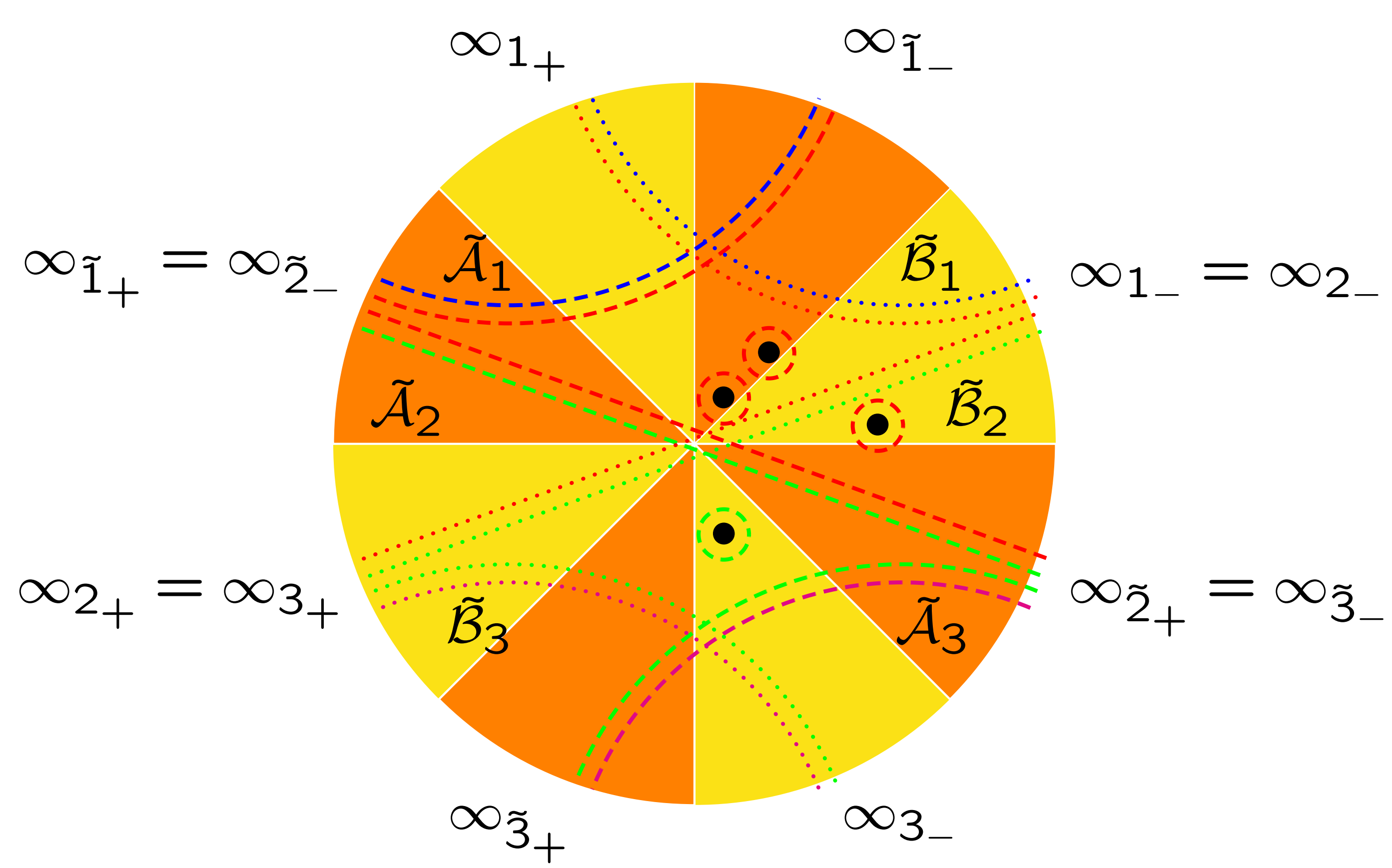
We then "protruding" integration contours to make them running between infinities "in pairs"



$$\oint_{\tilde{\mathcal{A}}_\alpha} f(x)dx \overset{\text{def}}{=} \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} \left(f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x})\right) dx + \sum_{s_i^{(\alpha\pm)}(\alpha)} \underset{}{res}\, f(\overset{\alpha_\pm}{x})$$

and

$$\oint_{\tilde{\mathcal{B}}_\alpha} f(x)dx \overset{\text{def}}{=} \int_{\infty_{\alpha_-}}^{\infty_{\alpha_+}} (f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x}))\, dx,$$



- Filling fractions

$$s_\alpha = \frac{1}{2i\pi}\oint_{\tilde{\mathcal{A}}_\alpha} \omega(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} \Big(\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x})\Big)dx, \quad \alpha = 1,\ldots,d.$$

The difference

$$\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x}) = \frac{\text{Wron}_{\alpha_+,\alpha_+}}{\psi_{\alpha_+}(x)\psi_{\alpha_-}(x)},$$

decreases exponentially in sectors where the both solutions ψ_{α_+} and ψ_{α_-} increase.

- First kind functions $v_k(\overset{\alpha}{x})$. Let h_k, $k = 1, \ldots, d-1$, be a basis of polynomials of degree $\leq d-2$. Then

$$v_k(\overset{\alpha}{x}) = \frac{1}{\hbar\, \psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} h_k(x')\, \psi_\alpha^2(x')\, dx'$$

with **the same** polynomial $h_k(x')$ for all the sheets and such that

$$I_{k,\alpha} = \oint_{\mathcal{A}_\alpha} v_k(x)\, dx = \delta_{k,\alpha} \quad k, \alpha = 1, \ldots, d-1;$$

$v_k(\overset{\alpha}{x})$ has double poles with no residue at the zeroes of ψ_α and behaves like $O(1/x^2)$ inside all the sectors including the sector S_α.

- The Riemann matrix of periods

$$\tau_{\alpha,i} \overset{\text{def}}{=} \oint_{\mathcal{B}_\alpha} v_i(x) dx$$

is symmetric [proof is not direct, however...]

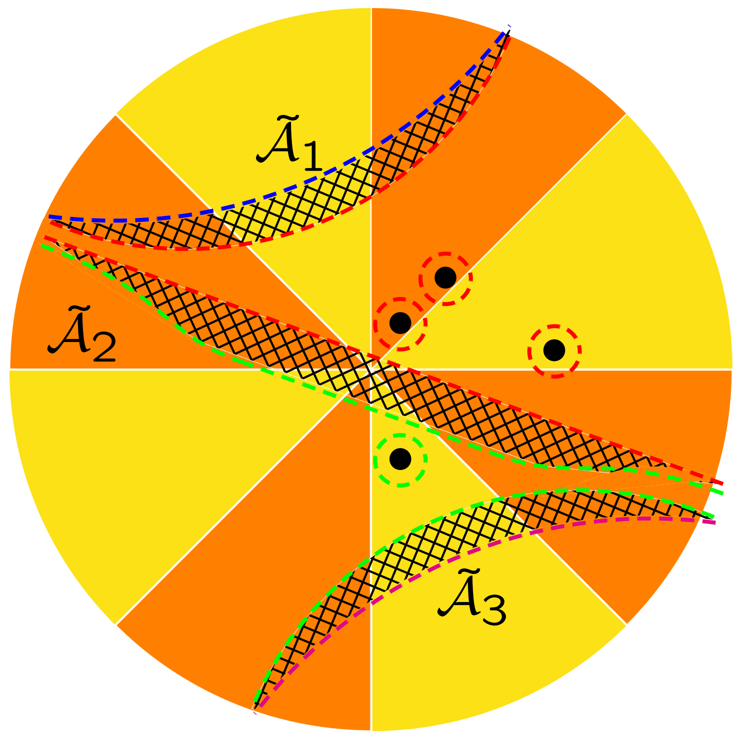
- The recursion kernel K

$$\widehat{K}(\overset{\alpha}{x}, z) = \frac{1}{\hbar}\frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x')\, \frac{dx'}{x'-z}$$

The recursion kernel $K(\overset{\alpha}{x}, z)$ reads

$$K(\overset{\alpha}{x}, z) = \widehat{K}(\overset{\alpha}{x}, z) - \sum_{j=1}^{d-1} v_j{}'(\overset{\alpha}{x}) C_j(z), \quad \hbar C_\alpha(z) = \oint_{\mathcal{A}_\alpha} \widehat{K}(x,z),\ \alpha = 1, \ldots, g$$

for z in the hatched domain



- Third kind differential: kernel $G(\overset{\alpha}{x},\overset{\beta}{z})$

$$G(\overset{\alpha}{x},\overset{\beta}{z}) = -\hbar\,\psi_\beta^2(z)\,\partial_z \frac{K(\overset{\alpha}{x},z)}{\psi_\beta^2(z)} = 2\hbar\frac{\psi'_\beta(z)}{\psi_\beta(z)}K(\overset{\alpha}{x},z) - \hbar\partial_z\,K(\overset{\alpha}{x},z)$$

is an analogue of $dE_{Q,\bar{Q}}(P)$, and

- The"quantum" Bergman kernel $B(\overset{\alpha}{x},\overset{beta}{z})$

$$B(\overset{\alpha}{x},\overset{beta}{z}) = -\frac{1}{2}\,\partial_z\,G(\overset{\alpha}{x},\overset{\beta}{z}).$$

$B(\overset{\alpha}{x},\overset{\beta}{z})$ is an analytical function of x with the double pole with zero residue at $x=z$ for $\alpha=\beta$, with double poles with zero residues in x and in z at the respective zeros $s_j^{(\alpha)}$ and $s_j^{(\beta)}$. The discontinuity along $\mathcal{A}$-cycles disappears upon differentiation, so $B(\overset{\alpha}{x},\overset{\beta}{z})$ is defined analytically in the whole complex plane.

The kernel B satisfies the loop equations:

$$(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x)\left(B(\overset{\alpha}{x},\overset{\beta}{z}) - \frac{1}{2(x-z)^2}\right) + \partial_z\,\frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} - \frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z} = P_2^{(0)}(x,\overset{\beta}{z})$$

- The properties of $B(\overset{\alpha}{x}, \overset{\beta}{z})$

- For every $\alpha = 1, \ldots, g$:

$$\oint_{\mathcal{A}_i} B(x, \overset{\beta}{z})\, dx = 0, \qquad \oint_{\mathcal{A}_j} B(\overset{\alpha}{x}, z)\, dz = 0;$$

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$$\oint_{\mathcal{B}_j} B(\overset{\alpha}{x}, z)\, dz = 2i\pi v_j(\overset{\alpha}{x});$$

- $B(\overset{\alpha}{x}, \overset{\beta}{z})$ is symmetric, $B(\overset{\alpha}{x}, \overset{\beta}{z}) = B(\overset{\beta}{z}, \overset{\alpha}{x})$.

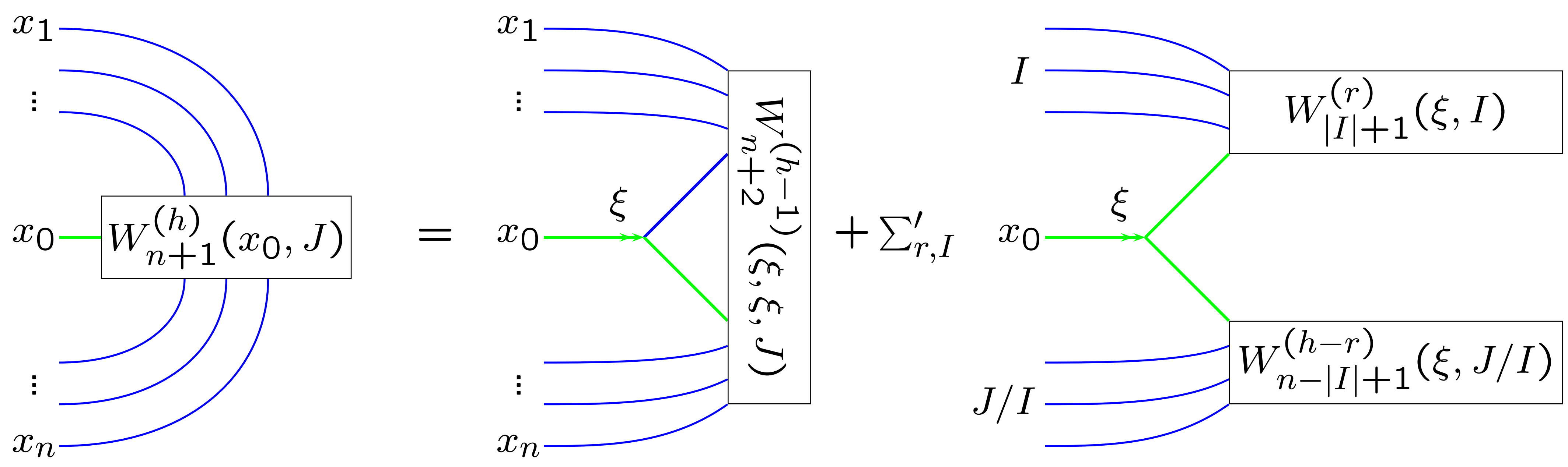
Corollary The period matrix $\tau_{k,\alpha}$ is symmetric:

$$\tau_{k,\alpha} = \oint_{\mathcal{B}_k} \oint_{\mathcal{B}_\alpha} B(z, x) dz\, dx.$$

We identify $B(\overset{\alpha}{x}, \overset{\beta}{z})$ with the two-point correlation function.

- Diagrammatic representation for correlation functions (solutions of the loop equation)

Recurrent relation:



$W_n^{(h)}(J)$ comprises all the diagrams with the corresponding automorphism factors (if any) such that

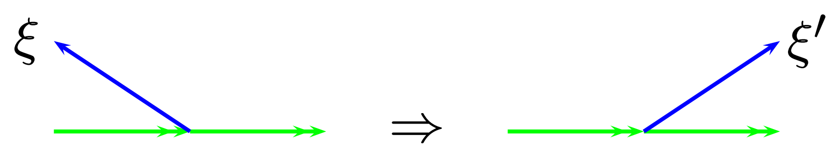
- for $W_n^{(h)}(J)$ it contains n external legs and h loops;
- we segregate one variable, say, x_1, and take all the maximum connected rooted subtrees starting at the vertex x_1 and not going to any other external leg;
- we associate the directed propagators $K(x,y)$ with all the edges of the rooted subtree; the direction is always from the root;
- all other propagators: h inner propagators and $n-1$ remaining external legs are $B(\overset{\alpha}{x},\overset{\beta}{y})$ if the vertices x and y are distinct and $B(\overset{\alpha_1}{x},\overset{\alpha_2}{x})$ for the loop composed from the single propagator;
- each rooted subtree establishes the partial ordering on the set of three-valent vertices of the diagram; we allow the inner propagators $B(x,y)$ to connect *only* the comparable vertices (a vertex is comparable to itself).

- Diagrammatic representation for symplectic invariants

Difficulty: to define $\beta\frac{\partial}{\partial\beta}$ for $|\Delta|^{2\beta}$ in the integral. Solution: instead of one-point function, we need **two-point functions**:

$$\hbar\frac{\partial}{\partial\hbar}W_n^{(h)}(J) = \int_{\mathcal{C}_{D_\xi}} d\xi \left[\int_\infty^\xi W_{n+2}^{(h-1)}(\bar{\xi}', \xi, J)d\xi' + \sum_{r=0}^{h}\sum_{I\subseteq J}\int_\infty^\xi W_{|I|+1}^{(r)}(\bar{\xi}', I)d\xi' \cdot W_{n-|I|+1}^{(h-r)}(\xi, J/I)\right],$$

where $\bar{\xi}$ must be taken to be an "innermost" variable; because $\int^\xi B(\xi', y) = G(\xi, y)$, we replace all the appearances



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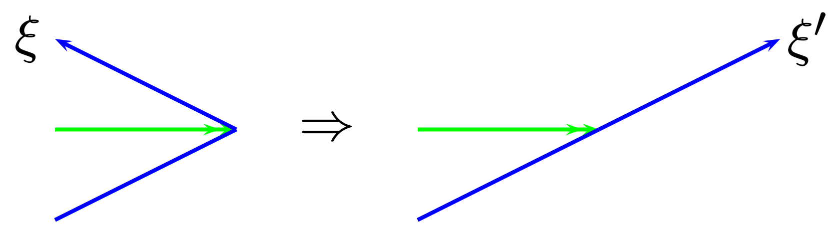
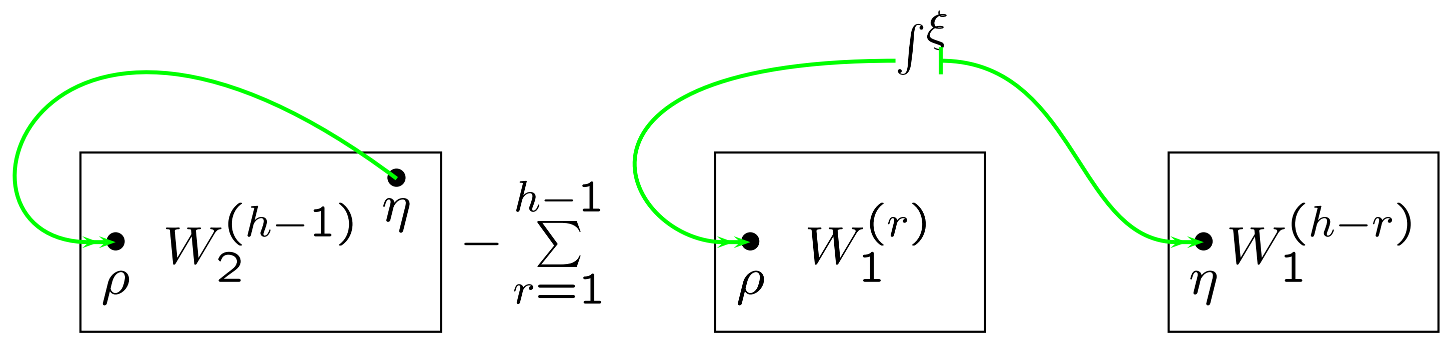


with *no additional factors*.

- The term $\mathcal{F}_h$

For the stable cases ($h \neq 0, 1$), the term $\mathcal{F}_h$ reads: Take all diagrams describing the stable terms $W_1^{(r)}(\bar{\xi})$ and $W_1^{(h-r)}(\xi)$ with $1 \leq r \leq h-1$ and $W_2^{(h-1)}(\xi, \bar{\xi})$, then

$$(2h-2)\mathcal{F}_h =$$



Here in the first term the vertices η and ρ are *distinct* (the ρ vertex is always the first three-valent vertex in the rooted tree) and joint by the propagator $K(\eta, \rho)$. In the second term, the integration over ξ survives ($\rho > \mathcal{C}_{D_\xi} > \eta$) and the symbol $\int^\xi$ indicates that we must insert the integration

$$\int_{\rho > \mathcal{C}_{D_\xi} > \eta} d\xi \int_{\infty_\alpha}^{\xi + \delta_\alpha} d\xi' K(\overset{\alpha}{\xi'}, \rho) K(\overset{\alpha}{\xi}, \eta)$$

between the integrations over the variables ρ and η.

Андрею Алексеевичу Славнову

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