Conformal Higher Spin Gauge Theory

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HS Symmetries

Unitary: $\varphi_{n_1 \dots n_s}$ - **rank** s **double traceless symmetric tensor** Fronsdal 1978

Gauge transformation:

$$\delta\varphi_{k_1 \dots k_s} = \partial_{(k_1}\varepsilon_{k_2 \dots k_s)} = 0\,, \qquad \varphi^{nm}{}_{nmk_5 \dots k_s} = 0\,, \qquad \varepsilon^{n}{}_{nk_1 \dots k_{s-3}} = 0$$

$\varepsilon_{k_1 \dots k_{s-1}}(x)$- **symmetric traceless tensor gauge parameter**

Conformal: $\phi_{n_1 \dots n_s}$ - **rank** s **traceless symmetric tensor** Fradkin-Tseytlin 1985

$$\delta\phi_{k_1 \dots k_s} = \Pi\partial_{(k_1}\varepsilon_{k_2 \dots k_s)} = 0\,, \qquad \phi^{n}{}_{nk_3 \dots k_s} = 0\,, \qquad \varepsilon^{n}{}_{nk_1 \dots k_{s-3}} = 0$$

Study of HS interactions is the search of symmetries beyond ad hoc geometric pictures.

Unfolded Dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

DOF = # of dynamical variables

Field theory: infinite number of DOF = spaces of functions

Maxwell $q \sim \overrightarrow{A}(x)$, $p \sim \overrightarrow{E}(x)$.

Covariant extension $t \to x^n$?

Unfolded dynamics: multidimensional generalization (1988)

$$\frac{\partial}{\partial t} \to d\,, \qquad q^i(t) \to W^\alpha(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W^\alpha_{n_1 \ldots n_p}(x)$$

a set of differential forms

Unfolded equations

$$dW^{\alpha}(x) = G^{\alpha}(W(x))\,, \qquad d = dx^n \partial_n$$

$G^{\alpha}(W)$: **function of "supercoordinates"** W^{α}

$$G^{\alpha}(W) = \sum_{n=1}^{\infty} f^{\alpha}{}_{\beta_1 \ldots \beta_n} W^{\beta_1} \wedge \ldots \wedge W^{\beta_n}$$

Covariant first-order differential equations

$d > 1$: **Nontrivial compatibility conditions:** $G^{\beta}(W) \wedge \frac{\partial G^{\alpha}(W)}{\partial W^{\beta}} = 0$ **equivalent to the generalized Jacobi identities**

$$\sum_{n=0}^{m} (n+1) f^{\gamma}{}_{[\beta_1 \ldots \beta_{m-n}} f^{\alpha}{}_{\gamma \beta_{m-n+1} \ldots \beta_m\}} = 0$$

Any solution to generalized Jacobi identities: FDA

Sullivan (1968), Auria and Fre (1982)

Vacuum Geometry

h: **a Lie algebra.** $\omega = \omega^\alpha T_\alpha$: **a 1-form valued in** h.

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^\alpha \wedge \omega^\beta [T_\alpha, T_\beta]$$

$$dW_0 + \frac{1}{2}[W_0\,, W_0] = 0\,,$$

Conformally flat background: $h = o(d, 2)$

Zero-curvature equations describe background geometry in a coordinate independent way.

BRST versus de Rham

R_α $(\alpha = 1, \ldots dim\, G)$ **right Lie vector fields on** G

$$[R_\alpha\,, R_\beta] = f_{\alpha\beta}{}^\gamma R_\gamma\,,$$

$$R_\alpha{}^b(x)\,\frac{\partial}{\partial x^b}\,R_\beta{}^a(x) - R_\beta{}^b(x)\,\frac{\partial}{\partial x^b}\,R_\alpha{}^a(x) \;= f_{\alpha\beta}{}^\gamma R_\gamma{}^a(x)\;.$$

Cartan forms

$$\omega^\alpha(x) = R^{-1}{}_a{}^\alpha(x) dx^a\,, \qquad d\omega^\alpha = -\frac{1}{2}\omega^\beta\omega^\gamma f_{\beta\gamma}{}^\alpha\,.$$

T **representation of** $\mathfrak{g}$

$$[T_\alpha\,, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma\,, \qquad [R_\beta\,, T_\alpha] = 0\,.$$

$$Q = c^\alpha(R_\alpha + T_\alpha) - \frac{1}{2}c^\alpha c^\beta b_\gamma f_{\alpha\beta}{}^\gamma\,, \qquad Q^2 = 0\,.$$

$$[c^\alpha\,, R_\beta] = 0\,, \qquad [b_\alpha\,, R_\beta] = 0\,, \qquad [c^\alpha\,, T_\beta] = 0\,, \qquad [b_\alpha\,, T_\beta] = 0$$

$$\{c^\alpha\,, b_\beta\} = \delta^\alpha_\beta\,, \qquad \{c^\alpha\,, c^\beta\} = 0\,, \qquad \{b_\alpha\,, b_\beta\} = 0\,,$$

Naive identification of c^α with ω^α fails because $[R_\alpha\,, c^\beta] = 0$ while $R_\alpha = R_\alpha{}^a(x)\frac{\partial}{\partial x^a}$ do not commute to $\omega^\alpha(x)$.

To proceed, redefine the notion of the vector fields

$$R_\alpha = R_\alpha{}^a(x)p_a\,, \qquad [p_a\,, f(x)] = \frac{\partial}{\partial x^a}f(x)\,, \qquad [p_a, p_b] = 0\,.$$

$$p_n = \frac{\partial}{\partial x^n} - c^\alpha\, b_\beta\, R^m_\alpha \frac{\partial}{\partial x^n}(R^{-1}{}_m{}^\beta).$$

p_n **is flat $Gl_{dim\,G}$ connection that has standard pure gauge form with $R_n{}^\alpha(x)$ as the gauge function.**

Now it is consistent to set $\omega^\alpha = c^\alpha$ since

$$[p_n\,,\omega^\alpha] = \frac{\partial}{\partial x^n}({R^{-1}}_m{}^\alpha(x))dx^m - c^\beta\,R^m_\beta\frac{\partial}{\partial x^n}({R^{-1}}_m{}^\alpha(x))\Big|_{c^\gamma=\omega^\gamma} = 0.$$

This gives

$$Q = \omega^\alpha(R_\alpha + T_\alpha) - \frac{1}{2}\omega^\alpha\omega^\beta b_\gamma f_{\alpha\beta}{}^\gamma = \omega^\alpha(R^n_\alpha\frac{\partial}{\partial x^n} + T_\alpha) = D\,, \qquad (1)$$

$$D = d + \omega^\alpha T_\alpha\,, \qquad d = dx^n\frac{\partial}{\partial x^n}\,.$$

$$Q^2 = 0 \qquad \Longleftrightarrow \qquad D^2 = 0\,.$$

$$Q\phi = 0 \qquad \Longleftrightarrow \qquad D\phi = 0\,.$$

For trivial T,

$$Q\,F(x,\,c) = G(x,\,c)\;\Longleftrightarrow\; dF(x,\,\omega) = G(x,\,\omega).$$

Application to conserved currents O.Gelfond, MV arxiv:1001.2585

Dynamical content via σ_-−cohomology

At the free field level unfolded equations are

$$\mathcal{D}_0 C = 0\,,$$

C **is some** $o(d,2)$**-module:** $o(d,2)$**-tensors in this talk.**

Let G : **be some diagonalizable grading operator in** V **such that its spectrum is bounded from below:** $V = V_0 \oplus V_1 \oplus \dots$

$$(D + \sigma_-)\, C(x) = 0, \qquad C \subset V$$

$$[G, D] = 0 \qquad [G, \sigma_-] = -\sigma_-$$

σ_- **decreases the grading and is algebraic**

$$\mathcal{D}_0{}^2 = 0\,, \qquad \Rightarrow\ \sigma_-{}^2 = 0\,, \qquad D^2 = 0\,, \qquad \{D, \sigma_-\} = 0$$

D **and** σ_- **form a bicomplex.**

Analysis of Bianchi identities = analysis of cohomology $H^p(\sigma_-, V)$

Let C be a $p-$form:

- 1. Differential gauge symmetry parameters (like diffeomorphisms) : $H^{p-1}(\sigma_-, V)$
- 2. Dynamical fields (like metric) in C: $H^p(\sigma_-, V)$
- 3. Nontrivial gauge invariant differential operators (like Einstein and Weyl tensors): $H^{p+1}(\sigma_-, V)$
- 4. Syzygies : $H^k(\sigma_-, V) \quad k > p+1$

Off-shell-system : $\mathcal{H}^{p+1} \equiv 0$: $\mathcal{D}_0 C = 0$: algebraic constraints

On-shell-system : $\mathcal{H}^{p+1} \neq 0$: $\mathcal{D}_0 C = 0$: PDE

Conformal Algebra

$$o(d,2): \qquad [T^{AB}\,, T^{CD}] = \eta^{BC}T^{AD} - \eta^{AC}T^{BD} - \eta^{BD}T^{AC} + \eta^{AD}T^{BC}$$

$A = (a, -, +)$: $a, b, \ldots = 0, \ldots d-1$

$$\eta^{+-} = 1\,, \qquad \eta^{++} = \eta^{--} = 0\,, \qquad \eta^{\pm a} = 0$$

Lorentz notations

$$T^{-a} = P^a\,, \qquad T^{+a} = K^a\,, \qquad T^{+-} = D\,, \qquad T^{ab} = L^{ab}$$

D **induces the grading**

$$[D\,, P^a] = -P^a\,, \qquad [D\,, K^a] = K^a\,, \qquad [D\,, T^{ab}] = 0.$$

Conformal Gravity

Conformal gravity can be described in terms of

$o(d,2)$ one-form connection

$$W(x) = \frac{1}{2} W_{AB}(x) T^{AB}\,, \qquad W_{AB}(x) = dx^{\underline{n}} W_{\underline{n}\,AB}(x)$$

and two-form curvature

$$R(x) = \frac{1}{2} R_{AB}(x) T^{AB}\,, \qquad R_{AB}(x) = dW_{AB}(x) + W_{AC}(x) \wedge W^{C}{}_{B}(x)\,.$$

In Lorentz components

$$W(x) = h^a(x) P_a + \frac{1}{2}\omega^{ab}(x) L_{ab} + f_a(x) K^a + b(x) D\,,$$

$$R(x) = R^a(x) P_a + \frac{1}{2} R^{ab}(x) L_{ab} + r_a(x) K^a + r(x) D$$

$\omega^{ab}(x)$ is Lorentz connection, $f_a(x)$ and $b(x)$ are gauge fields for special conformal transformations and dilatation, $h^a = dx^{\underline{m}} h_{\underline{m}}{}^a$ is the vielbein

$$R^a = dh^a + \omega^a{}_b \wedge h^b - b \wedge h^a \,,$$

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} - \mathbf{h^a \wedge f^b + h^b \wedge f^a} \,,$$

$$r = db + h^a \wedge f_a \,, \qquad r^a = df^a + \omega^a{}_b \wedge f^b + b \wedge f^a \,.$$

Conformal gauge transformations

$$\delta h^a = \mathcal{D}^L \epsilon^a - \epsilon^a{}_b h^b + \epsilon h^a - \epsilon^a b \,, \qquad \mathcal{D}^L \psi^a = d\psi^a + \omega^a{}_b \wedge \psi^b \,,$$

$$\delta \omega^{ab} = \mathcal{D}^L \epsilon^{ab} - h^a \tilde{\epsilon}^b + \epsilon^a f^b + h^b \tilde{\epsilon}^a - \epsilon^b f^a \,,$$

$$\delta b = d\epsilon + h^a \wedge \tilde{\epsilon}_a - \epsilon^a f_a \,, \qquad \delta f^a = \mathcal{D}^L \tilde{\epsilon}^a - \epsilon^a{}_b f^b - \epsilon f^a + \tilde{\epsilon}^a b \,,$$

$\epsilon^a(x)$, $\epsilon^{ab}(x)$, $\tilde{\epsilon}_a(x)$ and $\epsilon(x)$ **are gauge parameters of translations, Lorentz transformations, special conformal transformations and dilatations, respectively.**

Dynamical content of conformal gravity

$b = 0$ by special conformal transformations

$$R^a = 0 \quad \Rightarrow \quad \omega = \omega(h)$$

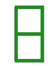
$$r = 0 \quad \Rightarrow \quad f_{\underline{nm}} - f_{\underline{mn}} = 0$$

Ricci part of R^{ab} equals zero: $f^a = f^a(h)$.

Nonzero components of the Lorentz curvature are in the Weyl tensor

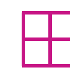
$$R_{ab}(x) = h^c \wedge h^d C_{ca,db}(x)$$

Trace and antisymmetric parts of the vielbein are pure gauge (Stueckelberg) with respect to local dilatations and local Lorentz transformations. The remaining differential gauge symmetry is associated with "local translations" = diffeomorphisms.

Start with one-form in $Y(1,1,0,\ldots|o(d,2)) =:$ 

Dynamical field: traceless symmetric tensor $Y(2,0\ldots|o(d-1,1)) =$ 

Gauge symmetry parameter: vector $Y(1,0\ldots|o(d-1,1)) =$ 

Gauge invariant field strength: Weyl tensor $Y(2,2,0,\ldots|o(d-1,1))=$ 

Physical content via supersymmetric mechanics

$$\sigma_- = \xi_a^\dagger T^{-a}\,, \qquad \sigma_-^\dagger = -\xi_a T^{+a}\,, \qquad \{\xi_a\,,\xi_b^\dagger\} = \eta_{ab}\,, \qquad \xi_a^\dagger = dx_a\,.$$

Supersymmetric Hamiltonian

$$\mathcal{H} = \{\sigma_-\,,\sigma_-^\dagger\} = \frac{1}{4}\Big(T^{L\,ab}T^L_{ab} - T^{AB}T_{AB}\Big) - \frac{1}{2}(\Delta + p)(\Delta + p - d)\,,$$

$$T^{AB}T_{AB} = -2\sum_{i=1}^{h} L_i(L_i + d + 2 - 2i)$$

$$\Delta = n_+ - n_-\,,$$

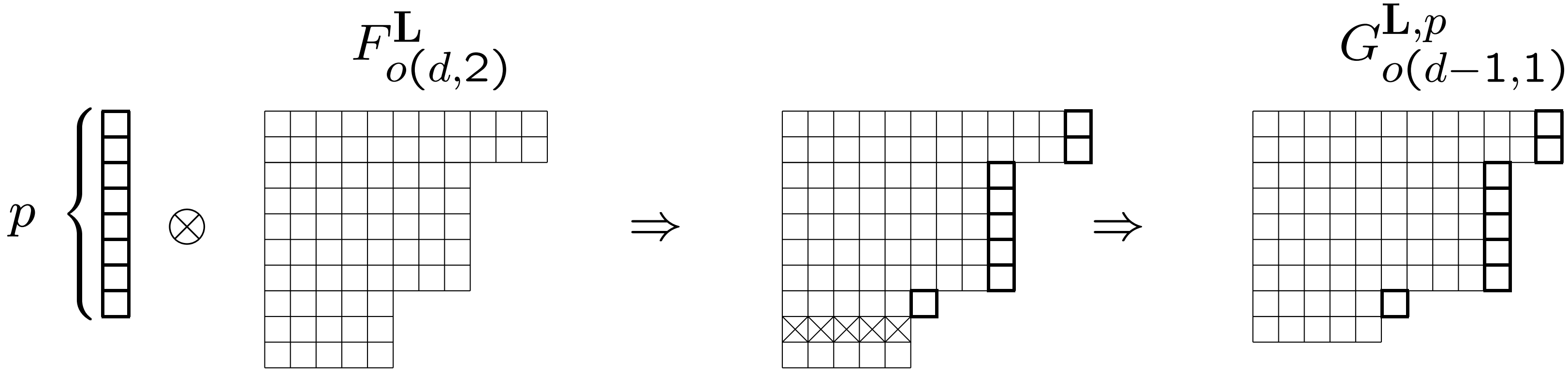
Dynamical fields, Weyl tensors and gauge symmetry parameters: supersymmetric vacua of $\mathcal{H}$

Mixed symmetry conformal fields

arxiv: 0909.5226

σ_- cohomology governs all mixed symmetry conformal fields in any d

p-form gauge field in $o(d,2)$-module $F^{\mathbf{L}}_{o(d,2)}$:



Dynamical field: $\phi^{dyn} \in G^{\mathbf{L},p}_{o(d-1,1)}$, $\quad \Delta(\phi^{dyn}) = -L_{p+1} + p$

Gauge symmetry parameter: $\varepsilon^{dif} \in G^{\mathbf{L},p-1}_{o(d-1,1)}$, $\quad \Delta\varepsilon^{dif} = -L_p + p - 1$

Gauge invariant Weyl tensor: $C(\phi^{dyn}) \in G^{\mathbf{L},p+1}_{o(d-1,1)}$, $\quad \Delta(C) = -L_{p+2} + p + 1$,

$$C(\phi^{dyn}) = \mathcal{L}_{p+1}\phi^{dyn}\,, \quad \delta\phi^{dyn} = \mathcal{L}_p\varepsilon^{dif}\,, \quad \mathcal{L}_{k+1}\mathcal{L}_k = 0 \quad \Longleftrightarrow \quad \delta(C(\phi^{dyn})) = 0$$

Differential operator $\mathcal{L}_k$ **has order** $q_k = L_k - L_{k+1} + 1$

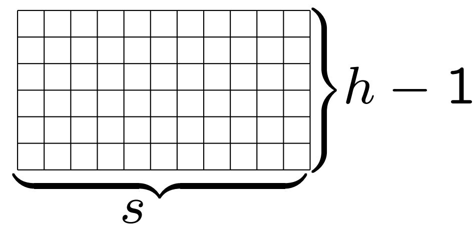
Example of Block

$$\mathbf{L} = (\underbrace{s-1,\dots,s-1}_{h},0,\dots 0\dots 0|o(d,2))$$

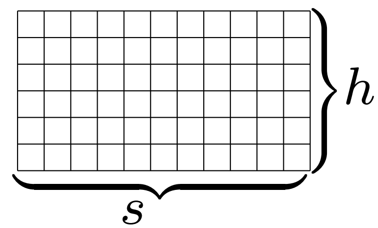
$p = h-1$ **Fradkin-Tseytlin case:** $h = 2, p = 1.$

Segal (2002): $p = 1, \forall h$; **Marnelius (2009):** $p = d/2-1\,,\qquad h = d/2$

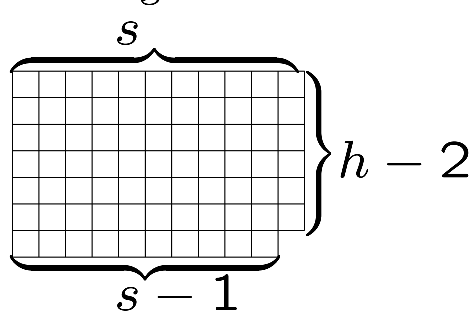
physical fields: ϕ^{dyn} :



Weyl tensor: C :



gauge parameter ε^{dif} :



Conformal actions

Any action

$$S = \int d^d x \mathcal{L}(E(C(\phi^{dyn})))$$

is gauge invariant.

Differential operator E is restricted by conformal invariance

Fradkin-Tseytlin action for $4d$ conformal HS fields:

$$S^s = \frac{1}{2} \int d^4 x \, C^{a_1 \dots a_s, b_1 \dots b_s}(\phi^{dyn}) C_{a_1 \dots a_s, b_1 \dots b_s}(\phi^{dyn}) \,, \qquad \phi^{dyn} = \phi_{a_1 \dots a_s} \,.$$

$s = 1$: **Maxwell theory,**

$s = 2$: **Conformal gravity**

Higher derivatives in higher dimensions

$$S^1 = \frac{1}{2} \int d^d x \Box^{\frac{d-4}{2}} C_{a,b} C^{a,b} \,.$$

Lagrangian

Conformal Lagrangian with $d-2\Delta$ derivatives for any Lorentz tensor conformal fields $\Psi_{1,2}$ characterized by the Young diagram $Y(l_1, l_2, \dots | o(d-1,1))$ and conformal dimension Δ

$$L = \sum_{m_l \geq 0} \frac{2^{\sum_{k=1}^{h} m_k}}{(q/2 - \sum_{k=1}^{h} m_k)!} \prod_{i=1}^{h} \frac{l_i!(l_i + \frac{(d+q)}{2} - i - 1 - \sum_{j=i}^{h} m_j)!}{m_i!(l_i - m_i)!(l_i + \frac{(d+q)}{2} - i - \sum_{j=i+1}^{h} m_j)!}$$

$$\partial_{a_1^1} \dots \partial_{a_{m_1}^1} \partial_{a_1^2} \dots \partial_{a_{m_2}^2} \dots \Psi_1^{a_1^1 \dots a_{m_1}^1 c_{m_1+1}^1 \dots c_{l_1}^1, a_1^2 \dots a_{m_2}^2 c_{m_2+1}^2 \dots c_{l_2}^2, \dots}$$

$$\Box^{q/2 - \sum_i m_i} \partial^{b_1^1} \dots \partial^{b_{m_1}^1} \partial^{b_1^2} \dots \partial^{b_{m_2}^2} \dots \Psi_{2\, b_1^1 \dots b_{m_1}^1 c_{m_1+1}^1 \dots c_{l_1}^1, b_1^2 \dots b_{m_2}^2 c_{m_2+1}^2 \dots c_{l_2}^2, \dots}$$

Coefficients are simple but terms may be linearly dependent because of Young symmetry properties

Primaries:

Dynamical fields ϕ^{dyn} both in the gauge and in the non-gauge cases

Weyl tensors $C(\phi^{dyn})$ in the gauge case

Conclusions

BRST analysis of closed forms:

Lagrangians and conserved currents

Nonunitary conformal theories:

to understand unitary HS theories with mixed fields in AdS_d via conformal

interpretation of AdS σ_- cohomology and actions.

FDA is universal if the generalized Jacobi identity holds for W interpreted as supercoordinates. HS FDAs are universal.

Every universal FDA = some L_∞ algebra

Equivalent form of compatibility condition

$$Q^2 = 0\,, \qquad Q = G^\alpha(W)\frac{\partial}{\partial W^\alpha}$$

Q-manifolds

Hamiltonian-like form of the unfolded equations

$$dF(W(x)) = Q(F(W(x))\,, \qquad \forall F(W)\,.$$

Invariant functionals: Q cohomology

$$S = \int L(W(x))\,, \qquad QL = 0 \qquad \textbf{(2005)}$$

The unfolded equation is invariant under the gauge transformation

$$\delta W^\alpha = d\varepsilon^\alpha + \varepsilon^\beta \frac{\partial G^\alpha(W)}{\partial W^\beta}$$

Properties

- General applicability
- Manifest (HS) gauge invariance
- Invariance under diffeomorphisms

  Exterior algebra formalism
- Interactions: nonlinear deformation of $G^{\alpha}(W)$
- Degrees of freedom are in 0-forms $C^i(x_0)$ at any $x = x_0$ (as $q(t_0)$) instead of phase coordinates in the Hamiltonian approach
- Natural realization of infinite symmetries with higher derivatives
- Lie algebra cohomology interpretation

Unfolding as a covariant twistor transform

Twistor transform

$$\begin{array}{ccc} & C(Y|x) & \\ {}^{\eta}\swarrow & & \searrow^{\nu} \\ M(x) & & T(Y)\,. \end{array}$$

$W^{\alpha}(Y|x)$ **are functions on the "correspondence space"** C.

Space-time M : **coordinates** x. **Twistor space** T : **coordinates** Y.

Unfolded equations: Penrose transform mapping functions on T **to solutions of field equations in** M.

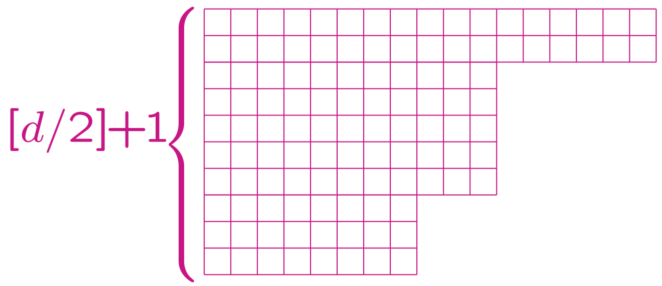
Independence of ambient space-time: Geometry is encoded by $G^{\alpha}(W)$

Physical dimension and metric emerge from unfolded equations 2002

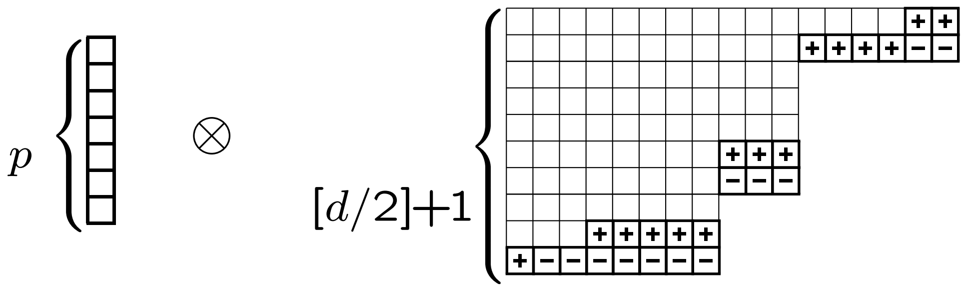
Physical space-times of different dimensions can coexist in an ambient space-time of higher (possibly infinite) **dimension.**

Branes are not localized while HS symmetries are unbroken

$o(d,2)$ content



Lorentz content



Theorem: $\mathcal{H} \geq 0$

Corollary: $\mathcal{H}(\mu_1) > \mathcal{H}(\mu_2)$ **implies** $\mathcal{H}(\mu_1) > 0$ **and hence** μ_1 **Lorentz diagrams do not contribute to the cohomology** H

This simple property immediately rules out most of the possibilities

For instance, no diagrams that contain both $+$ **and** $-$ **cells contribute.**

Primary conformal fields

Local special conformal transformation acts trivially: $\delta_{sc}\Psi = 0$

Global symmetry

$$\delta W_0^A(x) = \mathcal{D}_0 \epsilon^A(x) = 0$$

$$W_0 : \qquad h^a = dx^a\,, \qquad \omega_0^{ab} = 0\,, \quad f^a = 0\,, \quad b = 0\,.$$

$$\begin{aligned}
\tilde{\epsilon}_{0\,a}(x) &= \tilde{\varepsilon}_a \\
\epsilon_0(x) &= \varepsilon - x^a \tilde{\varepsilon}_a \\
\epsilon_0^{ab}(x) &= \varepsilon^{ab} + x^a \tilde{\varepsilon}^b - x^b \tilde{\varepsilon}^a \\
\epsilon_0^a(x) &= \varepsilon^a + x_b \varepsilon^{ab} - x^a \epsilon + x^a x^b \tilde{\varepsilon}_b - \frac{1}{2} x^2 \tilde{\varepsilon}^a
\end{aligned}$$

Global special conformal transformation acts via x-dependent dilatation and Poincarè transformation

Descendants are derivatives of Ψ

Oscillator realization

$$|\Psi(x)\rangle = \sum_{l_1\geq 0, l_2 \geq 0, \ldots} \frac{1}{\sqrt{l_1! l_2! \ldots}} \Psi^{a^1_1 \ldots a^1_{l_1}, a^2_1 \ldots a^2_{l_2} \ldots}(x) a^{\dagger 1}_{a^1_1} \ldots a^{\dagger 1}_{a^1_{l_1}} a^{\dagger 2}_{a^2_1} \ldots a^{\dagger 2}_{a^2_{l_2}} \ldots |0\rangle$$

$$[a^a_i \,, a^{\dagger j}_b] = \delta^a_b \delta^i_j \,, \qquad a^a_i |0\rangle = 0$$

$$t_{ij}|\Psi\rangle = 0 \,, \qquad t^i{}_j|\Psi\rangle = 0 \qquad j > i \,, \qquad (t^i{}_i - l_i)|\Psi\rangle = 0 \,, \quad \text{no summation over}$$

where

$$t^{ij} = a^{\dagger ai} a^{\dagger bj} \eta_{ab} \,, \qquad t^i{}_j = \frac{1}{2}\{a^{\dagger ai}\,, a^b_j\}\eta_{ab} \,, \qquad t_{ij} = a^a_i a^b_j \eta_{ab}$$

$$L = \langle \Psi_1 | E | \Psi_2 \rangle \,, \qquad E = E(\Box, \Theta_i) \,, \qquad \Box = \partial^a \partial_a \,, \qquad \Theta_i = a^{\dagger i\, a} a^b_i \partial_a \partial_b \,.$$

E :**order** q **differential operator**

$$E_a = [E\,, x_a] = \frac{\partial E(\partial)}{\partial \partial^a} \,, \qquad E^a \partial_a = qE \,.$$

Conformal invariance

Dilatation: $q = d - \Delta_1 - \Delta_2$

Special conformal transformation:

$$\delta L = \tilde{\varepsilon}^a \langle \Psi_1 | \left(-(q - 1 + \Delta_2) E_a + \frac{1}{2} E_b{}^b \partial_a + E^b M_{ba} \right) | \Psi_2 \rangle \,,$$

Ansatz:

$$E = \oint dt\, ds\, \sigma(s,t) \colon \exp t(\Box + \sum_{i=1}^{h} s^i \Theta_i) \colon ,$$

$$\oint dw w^{-1} = 1 \,, \qquad \oint dw w^n = 0 \,, \quad n \neq -1 \,, \qquad w = t \quad \text{or} \quad w = s^i .$$

Solution

$$\Delta_1 = \Delta_2 = \Delta = \frac{1}{2}(d - q) \,, \qquad \sigma(s,t) = t^{(\frac{q}{2} - 1)} \sigma(s)$$

$$\sigma(s) = \sum_{m_l \geq 0}^{\infty} (-2)^{\sum_{k=1}^{h} m_k} \prod_{i=1}^{h} \frac{(l_i + \frac{(d+q)}{2} - i - 1 - \sum_{j=i}^{h} m_j)!}{(l_i + \frac{(d+q)}{2} - i - \sum_{j=i+1}^{h} m_j)!} (s^1)^{-m_1 - 1} \dots (s^h)^{-m_h}$$

Relation between $L_1(\phi^{dyn})$ and $L_2(C(\phi^{dyn}))$?

a. If $q(L_2)=0$ then $L_1(\phi^{dyn})=L_2(C(\phi^{dyn}))$: $p=d/2-1$, $L_{d/2+1}=0$

b. If $q(L_2)>0$ then $L_2(C(\phi^{dyn}))=0$

Since C and ϕ^{dyn} can be treated as conformal fields with $p+1$ and p respectively, the Lagrangian for C treated as an independent field can have two types of conformal invariant equations:

a. $C=0$

b. $EC=0$, where E is some nontrivial differential operator

Since any nontrivial field equations on C (the option b.) are gauge invariant

$$EC(\phi^{dyn})=E\mathcal{L}_{p+1}\phi^{dyn}=0$$