Of Higgs, Unitarity and other Questions

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Abstract: On the verge of conclusive checks for the Standard Model by the LHC, I discuss some of the basic assumptions. The reason of this analysis stems from a recent proposal of an Electroweak Model based on a nonlinearly realized gauge group $SU(2) \otimes U(1)$, where, in the perturbative approximation, there is no Higgs boson. The model enjoys of the Slavnov-Taylor identities and therefore of the perturbative unitarity. On the other side it is commonly believed that the existence of the Higgs boson is entangled with the property of unitarity, when high energy processes are considered. The argument is based mostly on the Froissart bound and on the *Equivalence Theorem*. In this talk I briefly review some objections of mine on the validity of such arguments. Some open questions are pointed out in particular on the limit of zero mass for the vector mesons and on the fate of the longitudinal polarizations.

Introduction

The main assumptions for the construction of a massive YM local quantum field theory are

1. Renormalizability

2. Unitarity

3. Spontaneous Breakdown of Symmetry.

The mass is derived from the interaction with the Higgs field

$$S_{SSB} = S_{YM} + \frac{\Lambda^{D-4}}{g^2} \int d^D x \frac{1}{4} Tr\{|\partial_\mu \Omega - i A_\mu \Omega|^2\} + S_{BS}.$$

For $SU(2)$ **the matrix** Ω **may be parametrized by the real fields**

$$\Omega = \phi_0 + i\tau_i \phi_i, \quad \phi_0 = h + 2v, \quad \langle h \rangle = 0, \quad M = gv.$$

$W_L W_L$ elastic scattering

The attention has been focused on this process for different reasons. At high energy ($s, t >> M_W^2$) some anomalous behavior is expected for the longitudinal polarization. The idea is to entangle the presence of the Higgs boson to the requirement of unitarity. The calculations often make use of the so called Equivalence Theorem.

Part One: Unitarity

Unitarity: It is better to stress the conceptual difference between the Optical Theorem for the S-matrix

$$S = I\!I - iT, \quad S^\dagger S = I\!I, \quad \Longrightarrow ImT_{ii} \sim \sigma_{iT}. \tag{1}$$

and Perturbative Unitarity gives ($k > 0$)

$$0 = \sum_{j=0}^{k} S^{(j)\dagger} S^{(k-j)}. \tag{2}$$

where

$$S = \sum_{k=0}^{\infty} S^{(k)}, \quad S^{(0)} = I\!I. \tag{3}$$

For any finite order calculation $S_{in} = \sum_{j=0}^{k} S_{in}^{(j)}$

$$\sum_n |\sum_{j=0}^{k} S_{in}^{(j)}|^2 = \sum_n \sum_{l=0}^{k} \sum_{j=0}^{k} S_{in}^{(j)*} S_{in}^{(l-j)} + \sum_n \sum_{l=k+1}^{2k} \sum_{j=0}^{k} S_{in}^{(j+l)*} S_{in}^{(l-j)}$$

$$= 1 + \sum_n \sum_{l=k+1}^{2k} \sum_{j=l-k}^{k} S_{in}^{(j)*} S_{in}^{(l-j)}$$

There is always a violation of the Optical Theorem of order $\mathcal{O}(k+1)$**.**

The Optical Theorem has a meaning only if an operative definition of <u>forward</u> scattering exists. If long range interactions are present, then forward amplitude is an elusive object. Only eq. (2) has a meaning.

Part Two: Equivalence Theorem

BRST Transformations: **The BRST differential $\mathfrak{s}$ is obtained in the usual way by promoting the gauge parameters to the ghost fields c_a and by introducing the antighosts $\bar{c}_a$ coupled in a BRST doublet to the Nakanishi-Lautrup fields b_a:**

$$\mathfrak{s}\phi_a = \frac{1}{2}\phi_0 c_a + \frac{1}{2}\epsilon_{abc}\phi_b c_c\,, \qquad \mathfrak{s}\phi_0 = -\frac{1}{2}\phi_a c_a$$
$$\mathfrak{s}A_{a\mu} = (D_\mu[A]c)_a \quad , \mathfrak{s}\bar{c}_a = b_a, \quad \mathfrak{s}b_a = 0\,. \tag{4}$$

In the above equation $D_\mu[A]$ denotes the covariant derivative w.r.t. $A_{a\mu}$:

$$(D_\mu[A])_{ac} = \delta_{ac}\partial_\mu + \epsilon_{abc}A_{b\mu}\,. \tag{5}$$

The BRST transformation of c_a then follows by nilpotency

$$\mathfrak{s}c_a = -\frac{1}{2}\epsilon_{abc}c_b c_c\,. \tag{6}$$

The tree-level vertex functional is

$$
\begin{aligned}
\Gamma^{(0)} &= S_{YM} + \frac{\Lambda^{(D-4)}}{g^2}\, \mathfrak{s} \int d^D x \, (\bar{c}_a \partial A_a) \\
&\quad + \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \, (A^*_{a\mu} \mathfrak{s} A^\mu_a + \phi^*_a \mathfrak{s} \phi_a + \phi^*_0 \mathfrak{s} \phi_0 + c^*_a \mathfrak{s} c_a) \\
&= S_{YM} + \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \, (b_a \partial A_a - \bar{c}_a \partial_\mu (D^\mu[A] c)_a) \\
&\quad + \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \, (A^*_{a\mu} \mathfrak{s} A^\mu_a + \phi^*_0 \mathfrak{s} \phi_0 + \phi^*_a \mathfrak{s} \phi_a + c^*_a \mathfrak{s} c_a) \, . \qquad (7)
\end{aligned}
$$

In $\Gamma^{(0)}$ **we have also included the antifields** $A^*_{a\mu}, \phi^*_0, \phi^*_a$ **and** c^*_a **coupled to the nonlinear BRST variations of the quantized fields.**

Slavnov-Taylor Identity:

To simplify notations $b_a \to \frac{g^2}{\Lambda^{(D-4)}} b_a$**.**

The STI are: for the 1-PI functional (ZJ renormalization of composite operators)

$$\int d^D x \, (\Gamma_{A^*_{a\mu}} \Gamma_{A^\mu_a} + \Gamma_{\phi^*_a} \Gamma_{\phi_a} + \Gamma_{\phi^*_0} \Gamma_{\phi_0} + \Gamma_{c^*_a} \Gamma_{c_a} + b_a \Gamma_{\bar{c}_a}) = 0 \, , \qquad (8)$$

where we use the notation

$$\Gamma_X \equiv \frac{\delta \Gamma}{\delta X}.$$

While for the generating functional of the connected amplitudes one has

$$\int d^D x \, (- W_{A^*_{a\mu}} J_{a\mu} - W_{\phi^*_a} K_a - W_{\phi^*_0} K_0 + W_{c^*_a} \bar{\eta}_a - W_{b_a} \eta_a) = 0 \qquad (9)$$

where we use the notations

$$W_{A^*_{a\mu} \cdots} \equiv \frac{\delta^n W}{\delta A^*_{a\mu} \dots} = i^{n-1} \langle 0 | T((D^\mu[A]c)_a \dots) | 0 \rangle_C$$

for composite fields, while for elementary fields

$$W_{\underbrace{b_a \dots}_{n}} \equiv i^{n-1} \langle 0 | T(b_a \dots) | 0 \rangle_C .$$

External field sources

$$\int d^D x \, (J_{a\mu} A^\mu_a + K_a \phi_a + K_0 \phi_0 + \bar{c}_a \eta_a + \bar{\eta}_a c_a + J_{b_a} b_a) = 0 \, .$$

Landau Gauge Equation: **The equation associated to the gauge fixing gives**

$$\Gamma_{b_a} = \partial^\nu A_a^\mu \tag{10}$$

$$-J_{b_a} = \partial^\nu W_{A_a^\mu}. \tag{11}$$

The anti-ghost equation can be derived from eq. (49) and (9)

$$\Gamma_{\bar{c}_a} = \partial_\nu \Gamma_{A^* a\nu} \tag{12}$$

$$\eta = \partial^\nu W_{A_a^{*\nu}}. \tag{13}$$

From eqs. (9) and (13) one gets

$$\begin{aligned} W_{\phi^* \bar{c}} &= W_{\phi b} \\ W_{A_\mu^* \bar{c}} &= W_{A^\mu b} = -i\frac{p^\mu}{p^2}. \end{aligned} \tag{14}$$

Some Basic Results: **By a straightforward use of the above equations and of**

$$\Gamma W = -I\!I,$$

we get

$$W_{\phi b} = i\frac{p^{\nu}\Gamma_{\phi A^{\nu}}}{\Gamma_{\phi\phi}}\frac{1}{p^2}$$

$$(p^{\nu}\Gamma_{A^{\nu}\phi})^2 + p^2\Gamma_L\Gamma_{\phi\phi} = 0$$

$$W_{A^{\mu}\phi} = 0$$

$$W_L = 0.$$

$$W_{\phi\phi} = -\frac{1}{\Gamma_{\phi\phi}}.$$

Free Fields:

$$\Gamma_{A^\nu\phi} = iMp_\nu,\ \Gamma_{bb} = 0,\ \Gamma_{A^\nu b} = ip_\nu,$$
$$\Gamma_{\phi\phi} = p^2,\ \Gamma_{\phi b} = 0,\ \Gamma_L = M^2.$$

Then

$$W_{A^\mu\phi} = 0 \tag{15}$$

and

$$W_{A^\mu b} = -i\frac{p^\mu}{p^2}, \qquad W_{\phi b} = \frac{M}{p^2}$$
$$W_L = 0, \qquad W_{\phi\phi} = -\frac{1}{p^2}. \tag{16}$$

A Theorem: For $m \geq 1$

$$W_{b_{x_1}\cdots b_{x_m}} = 0 \ ,$$

$$W_{b_{x_1}\cdots b_{x_m}\phi^*_{z_1}\bar{c}_{y_1}\cdots\phi^*_{z_k}\bar{c}_{y_k}} = 0 \ ,$$

$$W_{b_x b_{x_1}\cdots b_{x_m}\phi_{w_1}\cdots\phi_{w_n}} = \sum_{i=1}^{n} W_{\phi^*_{w_i}\bar{c}_x b_1\cdots b_m\phi_{w_1}\cdots\overset{\vee}{\phi}_{w_i}\cdots\phi_{w_n}} \ ,$$

$$\sum_{j=1}^{k} (-)^j \ W_{b_{y_j}b_{x_1}\cdots b_{x_m}\phi^*_{z_1}\cdots\phi^*_{z_{k-1}}\bar{c}_{y_1}\cdots\overset{\vee}{\bar{c}}_{y_j}\cdots\bar{c}_{y_k}\phi_{w_1}\cdots\phi_{w_n}}$$

$$+\sum_{i=1}^{n} W_{b_{x_1}\cdots b_{x_m}\phi^*_{z_1}\cdots\phi^*_{z_{k-1}}\phi^*_{w_i}\bar{c}_{y_1}\cdots\bar{c}_{y_k}\phi_{w_1}\cdots\overset{\vee}{\phi}_{w_i}\cdots\phi_{w_n}} = 0 \qquad (17)$$

where $^\vee$ means omitted. Proof: just use STI.

b_a-insertions: **The quantity**

$$R \equiv i\frac{p^\nu \Gamma_{\phi A^\nu}}{M\Gamma_{\phi\phi}}|_{p^2=0} = \frac{p^2}{M} W_{b\phi}|_{p^2=0} \tag{18}$$

will appear all over again (at the tree level $R = 1$). The pole contribution gives

$$\begin{aligned} &\lim_{p^2=0} p^2 W_{b(p)\cdots} = (i\frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} W_{\widehat{\phi(p)}\cdots} + ip^\mu W_{\widehat{A^\mu(p)}\cdots})|_{p^2=0} \\ &= (-MR\Gamma_{\phi\phi} W_{\phi(p)\cdots} + ip^\mu W_{\widehat{A^\mu(p)}\cdots})|_{p^2=0}. \end{aligned} \tag{19}$$

Then one b_a insertion on a physical amplitude yields

$$\begin{aligned} &\lim_{p^2=0} p^2 W_{b(p)***} = (i\frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} W_{\widehat{\phi(p)}***} + ip^\mu W_{\widehat{A^\mu(p)}***})|_{p^2=0} \\ &= (-MR\Gamma_{\phi\phi} W_{\phi(p)***} + ip^\mu W_{\widehat{A^\mu(p)}***})|_{p^2=0} = 0, \end{aligned} \tag{20}$$

where the $***$ indicates all the other *physical* states obtained via reduction formulas.

More Notations: **The $\hat{}$ indicates that the external line (for instance attached to a A^μ) has been removed. According to this notation**

$$W_{A(p)BC\ldots} = \sum_X W_{A(p)X} W_{\widehat{X(p)}BC\ldots}. \tag{21}$$

The Longitudinal Polarization: The relation with the longitudinal polarization

$$\epsilon_L = \frac{E}{M|\vec{p}|}(\frac{\vec{p}^{\,2}}{E}, \vec{p}), \qquad E = \sqrt{M^2 + \vec{p}^{\,2}} \tag{22}$$

can be obtained by considering

$$\epsilon_L = \frac{E}{M|\vec{p}|}(|\vec{p}|, \vec{p}) - \frac{M}{E + |\vec{p}|}(1, \vec{0}). \tag{23}$$

It is usually assumed that

$$\epsilon_L = \frac{1}{M}(|\vec{p}|, \vec{p}) + \mathcal{O}(\frac{M}{E}) \tag{24}$$

gives the correct order of magnitude in the amplitudes

$$\epsilon_L^\mu W_{\widehat{A^\mu(p)}***}|_{p^2=M^2} = \frac{1}{M} p^\mu W_{\widehat{A^\mu(p)}***}|_{p^2=0} + \mathcal{O}(\frac{M}{E}), \tag{25}$$

then eq. (19) reads

$$\epsilon_L^\mu W_{\widehat{A^\mu(p)}***}|_{p^2=M^2} = iRW_{\widehat{\phi(p)}***}|_{p^2=0} + \mathcal{O}(\frac{M}{E}). \tag{26}$$

i.e. the statement of Lee, Quigg, Thacker (1977), Weldon (84), Chanowitz, Gaillard (1985), Gounaris, Kogerler, Neufeld (1986).

Unfortunately we will see that the evaluation of the order of magnitude given in eq. (18) cannot always be transfered to the amplitudes.

In particular there is a clear evidence that the limit $M = 0$ does not commute with the on-shell limit (reduction formula).

Two b_a Insertions: This is a very clear example of the singular behavior of the limit $M = 0$. The situation is somewhat different if we use eq. (19) or (25). We use first eq. (19) and subsequently we discuss the approach by exploiting eq.

(25).

NOTE: The insertion of a second b_a line is much simpler in the Landau gauge where $W_{A\phi} = 0$ remains valid beyond the tree approximation. In generic 't Hooft gauge there is a non-trivial mixing in the $\phi - \partial_\mu A^\mu$ space, which causes some important technical complexities.

One has

$$\lim_{p_1^2, p_2^2 = 0} i^2 p_1^\mu p_2^\nu W_{\widehat{A^\mu(p_1)}\,\widehat{A^\nu(p_2)}***} = \lim_{p_1^2, p_2^2 = 0} p_1^2 p_2^2 (W_{b_1 b_2 ***}$$
$$+ \frac{i p_1^\nu \Gamma_{\phi A^\nu}}{p_1^2} W_{\phi_1 b_2 ***} + \frac{i p_2^\mu \Gamma_{\phi A^\mu}}{p_2^2} W_{b_1 \phi_2 ***} + \frac{(i p_1^\nu \Gamma_{\phi A^\nu})}{p_1^2} \frac{(i p_2^\mu \Gamma_{\phi A^\mu})}{p_2^2} W_{\phi_1 \phi_2 ***}) \tag{27}$$

The first term is zero as in eq. (17).

The mixed terms can be obtained by performing the functional derivatives of the STI in eq. (9) with respect to η and K

$$W_{b_1\phi_2***} = W_{\phi_2^*\bar{c}_1***}. \tag{28}$$

Thus we get (with the use of $W_{\phi^*\bar{c}_1} = W_{b\phi}$)

$$\lim_{p_1^2,p_2^2=0} i^2 p_1^\mu p_2^\nu W_{\widehat{A^\mu(p_1)}\widehat{A^\nu(p_2)}***} = M^2R^2 \lim_{p_1^2,p_2^2=0} (\frac{\Gamma_{\phi_1\phi_!}}{p_1^2} p_2^2 W_{\bar{c}_2c_2} W_{\widehat{\bar{c}_1}\widehat{c_2}***}$$
$$+\frac{\Gamma_{\phi_2\phi_2}}{p_2^2} p_1^2 W_{\bar{c}_1c_1} W_{\widehat{\bar{c}_2}\widehat{c_1}***} + W_{\widehat{\phi_1}\widehat{\phi_2}***})$$
$$= M^2R^2 \lim_{p_1^2,p_2^2=0} (\bar{R} W_{\widehat{\bar{c}_1}\widehat{c_2}***} + \bar{R} W_{\widehat{\bar{c}_2}\widehat{c_1}***} + W_{\widehat{\phi_1}\widehat{\phi_2}***}), \tag{29}$$

where

$$\bar{R} = \lim_{p_1^2,p_2^2=0} \frac{\Gamma_{\phi_2\phi_2}}{p_2^2} p_1^2 W_{\bar{c}_1c_1}. \tag{30}$$

On the other side, if we consider multi b-field insertions by using eq. (19), where the scalar mode is replaced by the longitudinal mode according to eq. (25)

$$RW_{\widehat{\phi(p)}***} = \lim_{p^2=0} \frac{p^2}{M} W_{b(p)***} - i\epsilon^{\mu}_{L} W_{\widehat{A^{\mu}(p)}***}|_{p^2=M^2} + \mathcal{O}(\frac{M}{E}), \quad (31)$$

we get

$$\begin{aligned} R^2 W_{\widehat{\phi(p_1)}\widehat{\phi(p_2)}***} &= \lim_{p_1^2,p_2^2=0} \frac{p_1^2 p_2^2}{M^2} W_{b(p)***} \\ &+ i^2 \epsilon^{\mu_1}_{L} \epsilon^{\mu_2}_{L} W_{\widehat{A^{\mu_1}(p_1)}\widehat{A^{\mu_2}(p_2)}***}|_{p^2=M^2} + \mathcal{O}(\frac{M}{E}) \\ &= -\epsilon^{\mu_1}_{L} \epsilon^{\mu_2}_{L} W_{\widehat{A^{\mu_1}(p_1)}\widehat{A^{\mu_2}(p_2)}***}|_{p^2=M^2} + \mathcal{O}(\frac{M}{E}), \end{aligned} \quad (32)$$

where the mixed terms and the double b insertion are zero as required by eq. (17).

By replacing the scalar mode (unphysical) with the longitudinal polarization state, the value of the b-insertions changes in a substantial way.
We can conclude that the use of the substitution in eq. (19) is in contradiction with the results in eqs. (29) and (32).
Three b Insertions: **We consider three b insertions, which can be relevant in processes as $V + V \to l^+ + l^- + V$. We use once again the eq. (19) as in eq. (27)**

$$\lim_{p_1^2, p_2^2, p_3^2 = 0} \frac{i^3}{M^3} p_1^\mu p_2^\nu p_3^\rho W_{\widehat{A^\mu(p_1)}\,\widehat{A^\nu(p_2)}\,\widehat{A^\rho(p_3)}***} \tag{33}$$

$$
\begin{aligned}
&= \lim_{p_1^2,p_2^2,p_3^2=0} p_1^2p_2^2p_3^2(\frac{1}{M^3}W_{b_1b_2b_3***} + \frac{R\Gamma_{\phi_1\phi_1}}{M^2p_1^2}W_{\phi_1b_2b_3***} \\
&+\frac{R\Gamma_{\phi_2\phi_2}}{M^2p_2^2}W_{b_1\phi_2b_3***} + \frac{R\Gamma_{\phi_3\phi_3}}{M^2p_3^2}W_{b_1b_2\phi_3***} \\
&+\frac{1}{M}\frac{R\Gamma_{\phi_1\phi_1}}{p_1^2}\frac{R\Gamma_{\phi_2\phi_2}}{p_2^2}W_{\phi_1\phi_2b_3***} + \frac{1}{M}\frac{R\Gamma_{\phi_2\phi_2}}{p_2^2}\frac{R\Gamma_{\phi_3\phi_3}}{p_3^2}W_{b_1\phi_2\phi_3***} \\
&+\frac{1}{M}\frac{R\Gamma_{\phi_3\phi_3}}{p_3^2}\frac{R\Gamma_{\phi_1\phi_1}}{p_1^2}W_{\phi_1b_2\phi_3***} + \frac{R\Gamma_{\phi_1\phi_1}}{p_1^2}\frac{R\Gamma_{\phi_2\phi_2}}{p_2^2}\frac{R\Gamma_{\phi_3\phi_3}}{p_3^2}W_{\phi_1\phi_2\phi_3***})
\end{aligned} \tag{34}
$$

The mixed terms in eq. (34) are evaluated by using eq. (17).

$$\lim_{p_1^2,p_2^2,p_3^2=0} \frac{i^3}{M^3} p_1^\mu p_2^\nu p_3^\rho W_{\widehat{A^\mu(p_1)}\widehat{A^\nu(p_2)}\widehat{A^\rho(p_3)}***} = \lim_{p_1^2,p_2^2,p_3^2=0} p_1^2 p_2^2 p_3^2$$
$$(\frac{1}{M}\frac{R\Gamma_{\phi\phi}}{p_1^2}\frac{R\Gamma_{\phi\phi}}{p_2^2}W_{\phi_1\phi_2 b_3***} + \frac{1}{M}\frac{R\Gamma_{\phi\phi}}{p_2^2}\frac{R\Gamma_{\phi\phi}}{p_3^2}W_{b_1\phi_2\phi_3***}$$
$$+\frac{1}{M}\frac{R\Gamma_{\phi\phi}}{p_3^2}\frac{R\Gamma_{\phi\phi}}{p_1^2}W_{\phi_1 b_2\phi_3***} + \frac{R\Gamma_{\phi\phi}}{p_1^2}\frac{R\Gamma_{\phi\phi}}{p_2^2}\frac{R\Gamma_{\phi\phi}}{p_3^2}W_{\phi_1\phi_2\phi_3***})$$
$$= -R^3(\bar{R}W_{\widehat{\phi_1}\widehat{\bar{c}_2}\widehat{c_3}***} + \bar{R}W_{\widehat{\bar{c}_1}\widehat{\phi_2}\widehat{c_3}***} + \bar{R}W_{\widehat{\bar{c}_2}\widehat{\phi_3}\widehat{c_1}***}$$
$$+\bar{R}W_{\widehat{\phi_2}\widehat{\bar{c}_3}\widehat{c_1}***} + \bar{R}W_{\widehat{\bar{c}_3}\widehat{\phi_1}\widehat{c_2}***} + \bar{R}W_{\widehat{\phi_3}\widehat{\bar{c}_1}\widehat{c_2}***} + W_{\widehat{\phi_1}\widehat{\phi_2}\widehat{\phi_3}***}) \tag{35}$$

Four b Insertions **There is a surprising cancellation in the case of four b-insertions.**

$$\lim_{p_1^2\ldots p_4^2=0} p_1^{\mu}p_2^{\nu}p_3^{\sigma}p_4^{\rho}W_{\widehat{A_\mu}\widehat{A_\nu}\widehat{A_\sigma}\widehat{A_\rho}} = \lim_{p_1^2\ldots p_4^2=0} p_1^2p_2^2p_3^2p_4^2$$

$$(W_{b_1b_2b_3b_4} + MR\sum_j \frac{\Gamma_{\phi_j\phi_j}}{p_j^2}W_{b_{j+1}b_{j+2}b_{j+3}\phi_j}$$

$$+\frac{1}{2}M^2R^2\sum_{i\neq j}\frac{\Gamma_{\phi_k\phi_k}}{p_k^2}\frac{\Gamma_{\phi_l\phi_l}}{p_l^2}W_{b_ib_j\ \phi_k\ \phi_l}$$

$$+M^3R^3\sum_j \frac{\Gamma_{\phi_{j+1}\phi_{j+1}}}{p_{j+1}^2}\frac{\Gamma_{\phi_{j+2}\phi_{j+2}}}{p_{j+2}^2}\frac{\Gamma_{\phi_{j+3}\phi_{j+3}}}{p_{j+3}^2}W_{b_j\ \phi_{j+1}\ \phi_{j+2}\ \phi_{j+3}})$$

$$+M^4R^4\lim_{p_1^2\ldots p_4^2=0}W_{\widehat{\phi_1}\ \widehat{\phi_2}\ \widehat{\phi_3}\ \widehat{\phi_4}} \qquad (36)$$

Now according to the eq. (17) we have

$$W_{b_1 b_2 b_3 b_4} = 0 \tag{37}$$

and

$$W_{b_c b_b b_a \phi_1} = W_{b_c b_b \phi_1^* \bar{c}_a}. \tag{38}$$

A further use of eq. (17) tells that

$$W_{b_c b_b b_a \phi_1} = W_{b_c b_b \phi_1^* \bar{c}_a} = 0. \tag{39}$$

We deal with the term with one b-insertion before considering the most difficult term. We have again from eq. (17)

$$W_{b_j \ \phi_{j+1} \ \phi_{j+2} \ \phi_{j+3}} = \sum_{k=1,2,3} W_{\phi_{j+k}^* \ \bar{c}_j \ \phi_{j+k+1} \ \phi_{j+k+2}} \tag{40}$$

Thus the relevant term in eq. (36) becomes

$$\lim_{p_1^2 \ldots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 M^3 R^3 \sum_j \frac{\Gamma_{\phi_{j+1}\phi_{j+1}}}{p_{j+1}^2} \frac{\Gamma_{\phi_{j+2}\phi_{j+2}}}{p_{j+2}^2} \frac{\Gamma_{\phi_{j+3}\phi_{j+3}}}{p_{j+3}^2} W_{b_j\ \phi_{j+1}\ \phi_{j+2}\ \phi_{j+}$$

$$= \lim_{p_1^2 \ldots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 M^3 R^3 \sum_j \frac{\Gamma_{\phi_{j+1}\phi_{j+1}}}{p_{j+1}^2} \frac{\Gamma_{\phi_{j+2}\phi_{j+2}}}{p_{j+2}^2} \frac{\Gamma_{\phi_{j+3}\phi_{j+3}}}{p_{j+3}^2}$$

$$\sum_{k=1,2,3} W_{\phi^*_{j+k}\ \bar{c}_j\ \phi_{j+k+1}\ \phi_{j+k+2}}$$

$$= \lim_{p_1^2 \ldots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 M^3 R^3 \sum_j \frac{\Gamma_{\phi_{j+1}\phi_{j+1}}}{p_{j+1}^2} \frac{\Gamma_{\phi_{j+2}\phi_{j+2}}}{p_{j+2}^2} \frac{\Gamma_{\phi_{j+3}\phi_{j+3}}}{p_{j+3}^2}$$

$$\sum_{k=1,2,3} W_{\phi^*\ \bar{c}}(p_{j+k}) W_{\bar{c}c}(p_j) W_{\phi\phi}(p_{j+k+1}) W_{\phi\phi}(p_{j+k+2}) W_{\widehat{\bar{c}_{j+k}}\ \widehat{c_j}\ \widehat{\phi_{j+k+1}}\ \widehat{\phi_{j+k+2}}}$$

$$= M^4 R^4 \bar{R} \sum_j \sum_{k=1,2,3} W_{\widehat{\bar{c}_{j+k}}\ \widehat{c_j}\ \widehat{\phi_{j+k+1}}\ \widehat{\phi_{j+k+2}}} \qquad (41$$

Now we consider the most critical term in eq. (36). By using eq. (17) we get

$$W_{b_i b_j\ \phi_k\ \phi_l} = W_{\phi_k^*\ \bar{c}_i b_j\ \phi_l} + W_{\phi_l^*\ \bar{c}_i b_j\ \phi_k} \tag{42}$$

Unfortunately one cannot remove further the b insertion by using eq. (17). In the on-shell limit we re express b in terms of ϕ and $\partial^\mu A_\mu$ as in the single pole contribution of eq. (19). On-shell we have from eq. (42)

$$\lim_{p_j^2=0} p_j^2 W_{b_i b_j\ \phi_k\ \phi_l} = \lim_{p_j^2=0} [MR(W_{\phi_k^*\ \bar{c}_i\ \widehat{\phi_j}\ \phi_l} + W_{\phi_l^*\ \bar{c}_i\ \widehat{\phi_j}\ \phi_k})$$
$$+ig^2 p_j^\mu (W_{\phi_k^*\ \bar{c}_i\ \widehat{A_j^\mu}\ \phi_l} + W_{\phi_l^*\ \bar{c}_i\ \widehat{A_j^\mu}\ \phi_k})] \tag{43}$$

Thus the relevant terms in eq. (36) yield

$$\lim_{p_1^2\ldots p_4^2=0} p_1^2p_2^2p_3^2p_4^2\frac{1}{2}M^2R^2\sum_{i\neq j}\frac{\Gamma_{\phi_k\phi_k}}{p_k^2}\frac{\Gamma_{\phi_l\phi_l}}{p_l^2}W_{b_ib_j\ \phi_k\ \phi_l}$$

$$=\lim_{p_1^2\ldots p_4^2=0}\frac{1}{2}M^2R^2\sum_{i\neq j}p_i^2(\Gamma_{\phi_k\phi_k}\Gamma_{\phi_l\phi_l}[$$

$$MR(W_{\phi_k^*\ \bar{c}_i\ \widehat{\phi}_j\ \phi_l}+W_{\phi_l^*\ \bar{c}_i\ \widehat{\phi}_j\ \phi_k})$$

$$+ig^2p_j^\mu(W_{\phi_k^*\ \bar{c}_i\ \widehat{A_j^\mu}\ \phi_l}+W_{\phi_l^*\ \bar{c}_i\ \widehat{A_j^\mu}\ \phi_k})])$$

$$=\lim_{p_1^2\ldots p_4^2=0}\frac{1}{2}M^2R^2\sum_{i\neq j}p_i^2W_{c_i\bar{c}_i}[-MR(\Gamma_{\phi_k\phi_k}W_{b_k\phi_k}W_{\widehat{\bar{c}_k}\widehat{c_i}\ \widehat{\phi}_j\ \widehat{\phi_l}}$$

$$+\Gamma_{\phi_l\phi_l}W_{b_l\phi_l}W_{\widehat{\bar{c}_l}\widehat{c_i}\ \widehat{\phi}_j\ \widehat{\phi_k}})-ig^2p_j^\mu(\Gamma_{\phi_k\phi_k}W_{b_k\phi_k}W_{\widehat{\bar{c}_k}\widehat{c_i}\ \widehat{A_j^\mu}\ \widehat{\phi_l}}$$

$$+\Gamma_{\phi_l\phi_l}W_{b_1\phi_l}W_{\widehat{\bar{c}_l}\widehat{c_i}\ \widehat{A_j^\mu}\ \widehat{\phi_k}})] \tag{44}$$

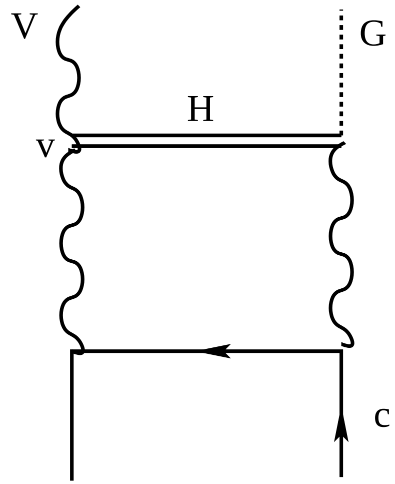
$$
= \lim_{p_1^2 \ldots p_4^2 = 0} \frac{1}{2} \bar{R} \sum_{i \neq j} [-M^4 R^4 (W_{\widehat{\bar{c}_k} \widehat{c_i} \widehat{\phi_j} \widehat{\phi_l}} + W_{\widehat{\bar{c}_l} \widehat{c_i} \widehat{\phi_j} \widehat{\phi_k}})
$$

$$
-ig^2 M^3 R^3 p_j^\mu (W_{\widehat{\bar{c}_k} \widehat{c_i} \widehat{A_j^\mu} \widehat{\phi_l}} + W_{\widehat{\bar{c}_l} \widehat{c_i} \widehat{A_j^\mu} \widehat{\phi_k}})]. \tag{45}
$$

The final result is then (eqs. (36), (37), (39), (41) and (44))

$$\lim_{p_1^2 \ldots p_4^2 = 0} p_1^\mu p_2^\nu p_3^\sigma p_4^\rho W_{\widehat{A_\mu}\widehat{A_\nu}\widehat{A_\sigma}\widehat{A_\rho}} = M^4 R^4 \lim_{p_1^2 \ldots p_4^2 = 0} W_{\widehat{\phi_1}\ \widehat{\phi_2}\ \widehat{\phi_3}\ \widehat{\phi_4}}$$
$$-i\frac{g^2}{2} \lim_{p_1^2 \ldots p_4^2 = 0} M^3 R^3 \bar{R} \sum_{i \neq j} p_j^\mu (W_{\widehat{\bar{c}_k}\widehat{c_i}\ \widehat{A_j^\mu}\ \widehat{\phi_l}} + W_{\widehat{\bar{c}_l}\widehat{c_i}\ \widehat{A_j^\mu}\ \widehat{\phi_k}}). \quad (46)$$

The second term in the RHS of eq. (46) is zero in the tree approximation. The dominant term at one loop is the box with two gauge-, one FP- and one Higgs boson-propagator. Three vertexes carry a single derivative. Then at high energy the behavior is $p'_\mu v \mathcal{O}(\frac{1}{s})$. Thus the total box contribution is $\sim M$, i.e. of the same order as the first term of the RHS.



Open Problems

- **What is the limit theory for $M = 0$, if any?**
- **In such a limit can we use v as the order parameter?**
- **How is the reshuffling of the physical modes? In particular does the Goldstone boson become a physical mode?**
- **The longitudinal mode ϵ_L is expected to become unphysical. How?**

We should give a second thought to results of Lee, Quigg, Thacker, Weldon, Chanowitz, Gaillard, Gounaris, Kogerler, Neufeld, Denner, Dittmaier, Hahn,... and look if there is some clue concerning the above listed questions. Maybe lattice simulations can help for the study of the transition to $M = 0$. These questions might be of great phenomenological significance.

As a conclusion I would dare to say that the above mentioned very distinguished physicists have extended too much the validity of there approximations (massless with $\mathcal{O}(\frac{M^2}{s})$).

Part Three: Nonlinearly Realized Gauge

Introduction

A common structure is present in the nonlinear sigma model (NLSM), in the massive Yang-Mills (YM) and in the Higgs-less Electro-Weak model (EW). For $SU(2)$ one has the action structures: NLSM (Ref. [1]-[6])

$$S_{NLSM} = \Lambda^{D-4}\frac{M^2}{4}\int d^Dx\; Tr\{\partial^\mu\Omega^\dagger\partial_\mu\Omega\}$$

the Stückelberg mass for YM (Ref. [7]-[8])

$$S_{YM} \sim \Lambda^{D-4}M^2\int d^Dx\; Tr\{[A_\mu - i\Omega\partial_\mu\Omega^\dagger]^2\}$$

and EW (Ref. [9]-[11]) mass terms

$$S_{EW} \sim \Lambda^{D-4}M^2\int d^Dx(\,Tr\,\{(gA_\mu - \frac{g'}{2}\Omega\tau_3B_\mu\Omega^\dagger - i\Omega\partial_\mu\Omega^\dagger)^2\}$$
$$+\frac{\kappa}{2}[Tr\{gA_\mu - \frac{g'}{2}\Omega\tau_3B_\mu\Omega^\dagger - i\Omega\partial_\mu\Omega^\dagger\tau_3\}]^2).$$

The $2 \times 2 \in SU(2)$ matrix may be parametrized by the real fields

$$\Omega = \phi_0 + i\tau_i\phi_i, \qquad \phi_0 = \sqrt{1 - \vec{\phi}^2}.$$

The constraint is implemented in the path integral measure

$$\prod_x \mathcal{D}^4\phi(x)\theta(\phi_0)\delta(\vec{\phi}(x)^2 + \phi_0^2(x) - 1) = \prod_x \mathcal{D}^3\phi(x)\frac{2}{\sqrt{1 - \vec{\phi}^2}}.$$

The non trivial measure in the path integral is the source of very interesting facts.

Renormalization? No. What else?

The non polynomial interaction makes the theory nonrenormalizable

$$S_{NLSM} = \Lambda^{D-4}\frac{M^2}{2}\int d^Dx\ \{\partial^\mu\phi_0\partial_\mu\phi_0 + \partial^\mu\vec{\phi}\partial_\mu\vec{\phi}\}$$
$$= \Lambda^{D-4}\frac{M^2}{2}\int d^Dx\ \{\partial^\mu\vec{\phi}\partial_\mu\vec{\phi} + \frac{1}{\phi_0^2}\ \phi_a\partial^\mu\phi_a\ \phi_b\partial_\mu\phi_b\}.$$

Vertexes carry second power of momenta, therefore already at one loop there is an infinite number of independent divergent amplitudes. Moreover it has been shown in the seventies and in the eighties that some divergences break chiral invariance (the global) at the same order.

Strategy: Abandon Hamiltonian formalism and do perturbation theory directly on the effective action functional Γ.

The Local Functional Equation (LFE)

The measure is invariant under "local left multiplication" transformations $\Omega \to U(\omega(x))\Omega$

$$
\delta\phi_0 = -\frac{\omega_a(x)}{2}\phi_a
$$

$$
\delta\phi_a = \frac{\omega_a(x)}{2}\phi_0 + \frac{\omega_c(x)}{2}\epsilon_{abc}\phi_b.
$$

Technical work to do: (i) find the algebra of operators closed under local left multiplication transformations by starting from the classical action, (ii) associate to every composite operator an external classical source (for subtraction strategy), (iii) write the LFE which follows from the invariance of the path integral measure.

Step (i)

This is simple in the NLSM. Introduce the "gauge field"

$$F_\mu = \frac{\tau_a}{2} F_{a\mu} \equiv i\Omega\partial_\mu\Omega^\dagger.$$

Its field strength tensor is zero (it describes a scalar mode) and its transformation properties are those of a gauge field

$$F_\mu \to U F_\mu U^\dagger + iU\partial_\mu U^\dagger.$$

The classical action can be written

$$S_{NLSM} = \Lambda^{D-4}\frac{M^2}{4}\int d^D x \; Tr\{F_\mu F^\mu\}.$$

Thus the closed set of operator is $\{\vec{\phi}, \phi_0, \vec{F}_\mu\}$

Step (ii)

The complete effective action at the tree level is then

$$\Gamma^{(0)} = \Lambda^{D-4} \int d^D x \, (\frac{M^2}{8}\{F_{a\mu} - J_{a\mu}\}^2 + K_0\phi_0).$$

The effective action $\Gamma[\vec{\phi}, \vec{J}_\mu, K_0]$ **is obtained via Legendre transform of the logarithm of the path integral functional**

$$Z[\vec{K}, \vec{J}_\mu, K_0] \equiv \int \prod_x \frac{2}{\phi_0} \mathcal{D}^3\phi(x) \exp[\Gamma^{(0)} + \int d^D y \vec{K}\vec{\phi}].$$

Step (iii)

Now we exploit the invariance of the path integral measure under local left multiplication $(\delta\phi_a = \frac{\omega_a(x)}{2}\phi_0 + \frac{\omega_c(x)}{2}\epsilon_{abc}\phi_b)$**.**

We expand for small parameter $\vec{\omega}(x)$ and obtain the LFE ($\langle \cdots \rangle$ indicates the mean over the weighted paths)

$$\int d^D x \left\langle (M_D^2 (F - J)_{a\mu} (\epsilon_{abc} \omega_c F_b^\mu + \partial^\mu \omega_a) \right.$$
$$\left. - \Lambda^{D-4} K_0 \frac{\omega_a}{2} \phi_a + \phi_0 K_a \frac{\omega_a}{2} + \epsilon_{abc} K_a \omega_c \phi_b)(x) \right\rangle = 0,$$

where

$$M_D^2 \equiv \Lambda^{D-4} M^2.$$

We will use the notation

$$\mathcal{D}[X]^\mu_{ab} = \delta_{ab} \partial_\mu - \epsilon_{abc} X_{c\mu}.$$

Thus for the effective action we get the local functional equation

$$-\partial^\mu \frac{\delta \Gamma}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta \Gamma}{\delta J_b^\mu} + \frac{\Lambda^{D-4}}{2} \phi_a K_0 + \frac{1}{2 \Lambda^{D-4}} \frac{\delta \Gamma}{\delta K_0} \frac{\delta \Gamma}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta \Gamma}{\delta \phi_b} = 0.$$

Hierarchy

The Spontaneous Breakdown of Symmetry is imposed by the condition

$$\left.\frac{\delta\Gamma}{\delta K_0}\right|_{\text{field \&sources}=0} = 1.$$

Then LFE naturally induces a strong hierarchy structure among the 1PI irreducible amplitudes: all amplitudes involving the $\vec{\phi}$ fields (descendant) are known in terms of the amplitudes involving only the (ancestor) sources $\vec{J}_\mu, K_0$. For instance, if we differentiate the LFE with respect to $J^\nu_{a'}(y)$ we get

$$\frac{M_D^2}{2}\partial^\mu \frac{\delta^2\Gamma}{\delta J^\mu_a(x)\delta J^\nu_{a'}(y)} + \frac{\delta^2\Gamma}{\delta\phi_a(x)\delta J^\nu_{a'}(y)} + 2\delta_{aa'}\partial_{x^\nu}\delta(x-y) = 0.$$

Weak Power Counting (WPC)

How many ancestor divergent amplitudes? The degree of divergence of a graph G for an ancestor amplitude is (n_L number of loops)

$$\delta(G) = D\ n_L - 2I + \sum_{j,k} j\ V_{jk} + N_F$$

$$n_L = I - \sum_{j,k} V_{jk} - N_F - N_{K_0} + 1$$

where I be the number of propagators, N_F the number of external F_μ sources and N_{K_0} those of K_0. V_{jk} denotes the number of vertexes with k ϕ-lines and j derivatives. The superficial degree of divergence $\delta(G)$ for a graph can be bounded by using standard arguments.

By removing I from these two equations one gets

$$\delta(G) = D\ n_L - 2n_L - \sum_{j,k}(2-j)\ V_{jk} - N_F - 2N_{K_0} + 2.$$

The classical action has vertexes with $j \leq 2$, therefore it can be stated that

$$\delta(G) \leq n_L(D-2) + 2 - N_F - 2N_{K_0}. \qquad (47)$$

For instance at $n_L = 1$ the only ancestor divergent (independent) amplitudes are $(J-J)$, $(J-J-J)$, $(J-J-J-J)$, $(K_0 - J - J)$, $(K_0 - K_0)$. The one-loop divergences of graphs where the descendant field appears ($\vec{\phi}$) are expressible all in terms of the ancestor divergences.

Perturbative Expansion

This is an *Ansatz*. Consider the generic dimension D. Start with $\Gamma^{(0)}$, read from it the value of the vertexes and construct $\Gamma^{(n)}$ for $n > 0$. The connected amplitudes $W^{(n)}$ can then be obtained. Few questions are in order

1. **Does $\Gamma^{(0)}$ obey the LFE? Yes, by construction**

2. **Does $\Gamma^{(n)}$, $n > 0$ obey the linearized LFE?**

$$\Big(-\partial^\mu \frac{\delta}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta}{\delta J_b^\mu} + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0}$$
$$+\frac{1}{2}\phi_0 \frac{\delta}{\delta \phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c \frac{\delta}{\delta \phi_b}\Big)\Gamma^{(n)} + \sum_{j=1}^{n-1} \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(j)}}{\delta \phi_a} \frac{\delta \Gamma^{(n-j)}}{\delta K_0} = 0.$$

3. **Assume that a *symmetric* subtraction procedure is given for the divergences in the limit $D = 4$, how is the breaking of the above equation?**

Answers

The answers to these questions are given in a compact form by the QAP

$$
\begin{aligned}
&\Big(- \partial^\mu \frac{\delta}{\delta J^\mu_a} + \epsilon_{abc} J^\mu_c \frac{\delta}{\delta J^\mu_b} \\
&- \frac{\Lambda^{D-4}}{2} K_0 \frac{\delta}{\delta K_a} + \frac{1}{2\Lambda^{D-4}} K_a \frac{\delta}{\delta K_0} + \epsilon_{acb} K_c \frac{\delta}{\delta K_b} \Big) Z \\
&= i \int \prod_x \frac{2}{\phi_0} \mathcal{D}^3\phi(x) [-\partial^\mu \frac{\delta\hat{\Gamma}}{\delta J^\mu_a} + \epsilon_{abc} J^\mu_c \frac{\delta\hat{\Gamma}}{\delta J^\mu_b} \\
&+ \frac{\Lambda^{D-4}}{2} \phi_a K_0 + \frac{1}{2\Lambda^{D-4}} \frac{\delta\hat{\Gamma}}{\delta K_0} \frac{\delta\hat{\Gamma}}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta\hat{\Gamma}}{\delta \phi_b}] \exp i [\hat{\Gamma} + \int d^D y \vec{K} \vec{\phi}],
\end{aligned}
$$

where $\hat{\Gamma}$ contains the counterterms $\hat{\Gamma}^{(j)}$

$$
\hat{\Gamma} = \Gamma^{(0)} + \sum_{j=1}^{\infty} \hat{\Gamma}^{(j)}.
$$

Subtraction Strategy

Thus if the counterterms at order n are missing, the linearized LFE is broken by the term

$$(-\partial^\mu \frac{\delta}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta}{\delta J_b^\mu} + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0}$$
$$+\frac{1}{2}\phi_0 \frac{\delta}{\delta \phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c \frac{\delta}{\delta \phi_b})\Gamma^{(n)} = -\frac{1}{2\Lambda^{D-4}} \sum_{j=1}^{n-1} \frac{\delta \hat{\Gamma}^{(j)}}{\delta K_0} \frac{\delta \hat{\Gamma}^{(n-j)}}{\delta \phi_a}.$$

Notice that $\frac{1}{\Lambda^{D-4}}\frac{\delta\Gamma^{(0)}}{\delta\phi_a}$ is independent from Λ^{D-4}. Thus we use Laurent expansion on

$$\Lambda^{-D+4}\Gamma^{(n)}$$

to define the finite part and the counterterm $\Lambda^{-D+4}\hat{\Gamma}^{(n)} = -\Lambda^{-D+4}\Gamma^{(n)}|_{\text{poles}}$.

Organization of the Divergences

The LFE is a power organizer of the divergences that WPC has classified. The full control can be obtained by finding the relevant local solutions of the linearized LFE

$$
\begin{aligned}
&(-\partial^\mu \frac{\delta}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta}{\delta J_b^\mu} + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0} \\
&+ \frac{1}{2}\phi_0 \frac{\delta}{\delta \phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c \frac{\delta}{\delta \phi_b}) \Gamma^{(n)}[\vec{\phi}, \vec{J}_\mu, K_0] = 0.
\end{aligned}
$$

This can easily achieved by using the technique of *bleaching*. **We shortly describe this procedure. The above equation naturally suggests the following infinitesimal transformations** δ_0

The Bleaching

The transformations

$$\delta_0 J^\mu_b = (\partial^\mu \delta_{ab} + \epsilon_{abc} J^\mu_c)\omega_a = \mathcal{D}[J]^\mu_{ba}\omega_a$$
$$\delta_0 F^\mu_a = \mathcal{D}[F]^\mu_{ab}\omega_b$$
$$\delta_0 K_0 = -\frac{\omega_a}{\Lambda^{D-4}}\frac{\delta\Gamma^{(0)}}{\delta\phi_a}$$
$$\delta_0(-\frac{\delta\Gamma^{(0)}}{\delta\phi_a}) = \Lambda^{D-4}\frac{1}{2}\omega_a K_0 + \frac{1}{2}\epsilon_{abc}\omega_c(-\frac{\delta\Gamma^{(0)}}{\delta\phi_b})$$

suggest the bleaching

$$\mathfrak{J}_\mu \equiv \Omega^\dagger(J_\mu - F_\mu)\Omega$$
$$\mathfrak{K}_0 \equiv \frac{K_0}{\phi_0} - \frac{M^2}{4}(J^\mu_b - F^\mu_b)\frac{\partial F_{b\mu}}{\partial\phi_a}\phi_a$$

Care in Bleaching

Few facts about bleaching. i) The relations are invertible, ii) In the case of $\mathfrak{J}_{a\mu}$, bleaching is a kind of gauge transformation where the parameters are the $\vec{\phi}$ fields

$$\mathfrak{J}_\mu = \Omega^\dagger J_\mu \Omega + i\Omega\partial_\mu\Omega$$
$$\partial_\mu \mathfrak{J}_\nu = \Omega^\dagger \left(\partial_\mu + \Omega\partial_\mu\Omega^\dagger\right)(J_\nu - F_\nu)\Omega = \Omega^\dagger \mathcal{D}_\mu[F](J_\nu - F_\nu)\Omega$$

iii) the invariants can be constructed by using $\mathfrak{J}_\mu$ and $\mathfrak{K}_0$ and their space-time derivatives. Ancestor amplitudes do not depend explicitly from $\vec{\phi}$. We consider only those relevant for the one-loop divergences.

The one-loop Invariants

$$\mathcal{I}_1 = \int d^Dx\,[D_\mu(F-J)_\nu]_a[D^\mu(F-J)^\nu]_a\,,$$
$$\mathcal{I}_2 = \int d^Dx\,[D_\mu(F-J)^\mu]_a[D_\nu(F-J)^\nu]_a\,,$$
$$\mathcal{I}_3 = \int d^Dx\,\epsilon_{abc}[D_\mu(F-J)_\nu]_a(F_b^\mu - J_b^\mu)(F_c^\nu - J_c^\nu)\,,$$
$$\mathcal{I}_4 = \int d^Dx\,(\frac{K_0}{\phi_0} + \frac{M^2}{4}[F_b^\mu - J_b^\mu]\frac{\partial F_{b\mu}}{\partial\phi_a}\phi_a)^2\,,$$
$$\mathcal{I}_5 = \int d^Dx\,(\frac{K_0}{\phi_0} + \frac{M^2}{4}[F_b^\mu - J_b^\mu]\frac{\partial F_{b\mu}}{\partial\phi_a}\phi_a)(F_c^\mu - J_c^\mu)^2\,,$$
$$\mathcal{I}_6 = \int d^Dx\,(F_a^\mu - J_a^\mu)^2(F_b^\nu - J_b^\nu)^2\,,$$
$$\mathcal{I}_7 = \int d^Dx\,(F_a^\mu - J_a^\mu)(F_a^\nu - J_a^\nu)(F_{b\mu} - J_{b\mu})(F_{b\nu} - J_{b\nu})\,,$$

where D_μ denotes the covariant derivative w.r.t $F_{a\mu}$:

$$D_{ab\mu} = \delta_{ab}\partial_\mu - \epsilon_{abc}F_{c\mu}\,.$$

The Counterterms

The counterterms are evaluated by extracting the pole parts from the relevant amplitudes given by the effective action functional normalized by $\Lambda^{-D+4}\Gamma$. It is very important to care about the relation

$$2(\mathcal{I}_1 - \mathcal{I}_2) - 4\mathcal{I}_3 + (\mathcal{I}_6 - \mathcal{I}_7) = \int d^D x \mathcal{G}_{a\mu\nu}[\mathfrak{J}]\mathcal{G}_a^{\mu\nu}[\mathfrak{J}] = \int d^D x \mathcal{G}_{a\mu\nu}[J]\mathcal{G}_a^{\mu\nu}[J] =\sim 0.$$

The right hand term is sterile: no descendant terms are generated. Now the calculation gives

$$\Gamma^{(1)} = \frac{1}{D-4}\frac{\Lambda^{D-4}}{(4\pi)^2}[-\frac{1}{12}(\mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3) + \frac{1}{48}(\mathcal{I}_6 + 2\mathcal{I}_7)$$
$$+\frac{3}{2}\frac{1}{M^4}\mathcal{I}_4 + \frac{1}{2}\frac{1}{M^2}\mathcal{I}_5]$$

The massive Yang-Mills

Ω **describes the Goldstone bosons, that are here unphysical modes. Then it is important to ensure that the Slavnov-Taylor Identity (STI) is valid in order to preserve unitarity. The LFE must be compatible with the STI. A suitable gauge-fixing term will help to achieve this result. The Landau gauge is the simplest, since the tadpole contributions can be neglected in most cases. The transformations to be considered are the local left** $SU(2)_L$ **and the global right** $SU(2)_R$ **on** Ω**, the gauge fields** A_μ**, the Faddeev-Popov fields** $c, \bar{c}$**. Few external sources are needed in order to describe the complete (under the** $SU(2)_L \otimes SU(2)_R$**) set of composite operators.**

Landau Gauge-fixing

$$\Gamma^{(0)} = S_{YM} + \frac{\Lambda^{D-4}}{g^2}\int d^Dx\,(B_a(D^\mu[V](A_\mu - V_\mu))_a - \bar{c}_a(D^\mu[V]D_\mu[A]c)_a)$$
$$+ \int d^Dx\,(A^*_{a\mu}sA^\mu_a + \phi^*_0 s\phi_0 + \phi^*_a s\phi_a + c^*_a sc_a + K_0\phi_0)\,.$$

$$S_{YM} = \frac{\Lambda^{(D-4)}}{g^2}\int d^Dx\,(-\frac{1}{4}G_{a\mu\nu}[A]G^{\mu\nu}_a[A] + \frac{M^2}{2}(A_{a\mu} - F_{a\mu})^2)\,.$$

$$\Omega = \frac{1}{v}(\phi_0 + i\tau_a\phi_a), \qquad \phi_0^2 + \phi_a^2 = v^2 \tag{48}$$

where v is a parameter with dimension equal one. We stress that v is not a parameter of the model, because it can can be removed by a rescaling of the fields $\vec{\phi}, \phi_0$.

STI

The Slavnov-Taylor Identity necessary for the validity of physical unitarity: S-matrix satisfies the following equation at the perturbative level

$$\langle \alpha | \beta \rangle = \sum_{n \in \{\text{physical states}\}} \langle \alpha | S | n \rangle \langle n | S^\dagger | \beta \rangle$$

if both α, β are physical states. This in general is valid if the Slavnov-Taylor identity is valid.

$$\mathcal{S}(\Gamma) = \int d^D x \, \Big(\frac{\delta \Gamma}{\delta A^*_{a\mu}} \frac{\delta \Gamma}{\delta A^\mu_a} + \frac{\delta \Gamma}{\delta \phi^*_a} \frac{\delta \Gamma}{\delta \phi_a} + \frac{\delta \Gamma}{\delta c^*_a} \frac{\delta \Gamma}{\delta c_a} + B_a \frac{\delta \Gamma}{\delta \bar{c}_a} - K_0 \frac{\delta \Gamma}{\delta \phi^*_0} \Big) = 0 \, .$$

Massive YM: the Local Functional Equation

$$\mathcal{W}(\Gamma) \equiv \int d^D x \alpha_a^L(x)(-\partial_\mu \frac{\delta\Gamma}{\delta V_{a\mu}} + \epsilon_{abc} V_{c\mu} \frac{\delta\Gamma}{\delta V_{b\mu}} - \partial_\mu \frac{\delta\Gamma}{\delta A_{a\mu}}$$
$$+\epsilon_{abc} A_{c\mu} \frac{\delta\Gamma}{\delta A_{b\mu}} + \epsilon_{abc} B_c \frac{\delta\Gamma}{\delta B_b} + \frac{1}{2} K_0 \phi_a + \frac{1}{2} \frac{\delta\Gamma}{\delta K_0} \frac{\delta\Gamma}{\delta \phi_a}$$
$$+\frac{1}{2}\epsilon_{abc}\phi_c \frac{\delta\Gamma}{\delta \phi_b} + \epsilon_{abc} \bar{c}_c \frac{\delta\Gamma}{\delta \bar{c}_b} + \epsilon_{abc} c_c \frac{\delta\Gamma}{\delta c_b}$$
$$+\epsilon_{abc} A^*_{c\mu} \frac{\delta\Gamma}{\delta A^*_{b\mu}} + \epsilon_{abc} c^*_c \frac{\delta\Gamma}{\delta c^*_b} + \frac{1}{2}\phi^*_0 \frac{\delta\Gamma}{\delta \phi^*_a}$$
$$+\frac{1}{2}\epsilon_{abc}\phi^*_c \frac{\delta\Gamma}{\delta \phi^*_b} - \frac{1}{2}\phi^*_a \frac{\delta\Gamma}{\delta \phi^*_0}) = 0 \, .$$

Γ also obeys the Landau gauge equation

$$\frac{\delta\Gamma}{\delta B_a} = \frac{\Lambda^{D-4}}{g^2} D^\mu[V](A_\mu - V_\mu)_a \qquad (49)$$

Linearized Equations and Induced Transformations

The structure of both STI and LFE is standard. Thus we can

1. **Establish the full hierarchy (only the Goldstone bosons are descendant fields)**
2. **Confirm the validity of the WPC**
3. **Introduce the linearized STI and LFE**
4. **Extract from the linearized STI and LFE the generators of the transformations on the effective action Γ**
5. **Check that the generators stemming from STI commute with those from LFE**

Subtraction procedure

With these tools we can construct the most general classical action compatible with the WPC and the invariance under the BRST transformations and the LFE induced symmetry. Surprisingly enough the resulting action is the standard YM field theory with a Stückelberg mass term.

The subtraction procedure of the divergences is then the same as in the NLSM: subtraction of the pure pole parts in the Laurent expansion around $D = 4$ of the normalized amplitudes $\Lambda^{-D+4}\Gamma$. This subtraction procedure has been implemented in the one-loop calculation of the gauge field two-point function.

Consistency of the Subtraction Procedure

The two-loop self-energy amplitude have been considered from the point of view of the consistency. It has been argued that the subtraction scheme is consistent: i) the counterterms are local ii) physical unitarity is satisfied iii) the STI and LFE induced symmetry on Γ is preserved. Ref. [8]

1) Explicit calculation of the gauge field two-point function.

2) Check that the counterterms are local at the two-loop level.

3) Validity of unitarity.

4) All divergences (infinite) at the one-loop level are subtracted by a finite number of counterterms.

Outlook and (some) open questions

- **Phenomenological applications (Andrea Quadri's talk)**
- **Running constant (dependence from Λ)**
- **How to proceed with a generic regularization tool?**
- **Well-defined strategy of minimal subtraction with anti-commuting γ_5.**
- **Extension to Grand Unified groups**

Conclusions

LFE $\Longrightarrow$

- **Hierarchy**
- **WPC**
- **Consistent subtraction procedure (symmetric and local)**
- **Finite number of physical parameters**

For massive YM, FTI and Landau gauge equation $\Longrightarrow$

- **Physical unitarity**
- **Consistency with LFE.**

't Hooft-Feynman gauge possible (many tadpole diagrams).

[1] **hep-th/0504023 Nonlinear σ model**

[2] **hep-th/0506220 A Weak Power-Counting Theorem**

[3] **hep-th/0511032 A two-loop example in Nonlinear σ model**

[4] **hep-th/0611063 Large Mass Limit of the Linear Sigma Model**

[5] **hep-th/0701197 Comments on the Symmetric Subtraction**

[6] **hep-th/0701212 Path-integral over non-linearly realized groups and Hierarchy solutions**

[7] **0705.2339 [hep-th] Massive Yang-Mills Theory**

[8] 0709.0644 [hep-th] One-loop Self-energy in a Massive Yang-Mills

[9] 0807.3882 [hep-ph] SU(2) X U(1) Electroweak Model. I

[10] 0809.1994 [hep-th] SU(2) x U(1) Electroweak Model. II

[11] 0903.0281 [hep-th] One-loop Self-energies in the Electroweak Model

[12] 0907.0426 [hep-th] Free complex filed