An Analytic Result for the Two-Loop Hexagon Wilson Loop in $N = 4$ SYM

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- Historiographical introduction. Scattering amplitudes and Wilson loops.
- Evaluating the two-loop hexagon Wilson loop.
- Results and perspectives.

Solvable models in 4 dimensions?

$N = 4$ SUSY YM
It is unphysical but has indirect connections with QCD

Soft anomalous dimensions at 3 loops in $N = 4$ SUSY YM
[Kotikov, Lipatov, Onishchenko and Velizhanin'04]

$\longleftrightarrow$

leading-transcendentality part of three-loop soft anomalous dimension in QCD [Moch, Vermaseren and Vogt'04]

Planar limit (LO in $1/N_c$)

AdS/CFT correspondence. Maldacena conjecture.

Maximally-helicity violating (MHV) amplitudes

The colour-stripped two-loop scattering amplitude with an arbitrary number n of external legs

$$M_n^{(L)} = M_n^{(0)} \; m_n^{(L)}$$

In two loops: [Anastasiou, Bern, Dixon and Kosower'03]

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + C^{(2)} + O(\epsilon)$$

where

$C^{(2)} = -\zeta(2)^2/2$, $f^{(2)}(\epsilon) = -\zeta(2) - \zeta(3)\epsilon - \zeta(4)\epsilon^2$

$\epsilon = (4-d)/2$

A conjecture: in L loops: [Bern, Dixon and Smirnov'05]

$$\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon)$$

$$= \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right] .$$

where $a = \frac{g^2 N}{8\pi^2}$ and $f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$.

The exponential conjecture (Ansatz) is correct for

three-loop four-point amplitude [BDS'05]
two-loop five-point amplitude [Bern et al.'06; Cachazo et al.'06]

$n = 6$?

AdS/CFT correspondence $\rightarrow$ in the strong-coupling limit the ansatz must break down for large n [Alday and Maldacena'07]

The computation of the two-loop six-edged Wilson loop:

[Drummond, Henn, Korchemsky and Sokatchev '07]

"either the ansatz or the duality relation between amplitudes and Wilson loops were to break down for two-loop six-point amplitudes"

Analysis of the multi-Regge kinematics in a Minkowski region [Bartels, Lipatov and Sabio Vera'08; Schabinger'09]

The remainder

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - C^{(2)}$$

Two-loop numerical calculations for $n = 6$:

[Bern et al.'08; Cachazo et al.'06]

$$R_6^{(2)} \neq 0$$

Further numerical analysis at $n = 6, 7, 8$ [Anastasiou et al.'09]

$$R_6^{(2)} =?$$

Duality between amplitudes and Wilson loops in the strong-coupling limit [Alday and Maldacena'07]

The duality holds for MHV amplitudes for general coupling [Drummond, Henn, Korchemsky and Sokatchev '07] because of the dual conformal symmetry of both the amplitudes and the Wilson loops

Checking duality at weak coupling:

[Drummond et al.'07–08, Brandhuber et al.'07]

one loop, general n and two loops, $n = 4, 5, 6$

L-loop light-like Wilson loop exhibits a conformal symmetry, and that the solution of the Ward identity for a special conformal boost is the ansatz, augmented, for $n \geq 6$, by a function $R^{(L)}_{n,WL}$ of conformally invariant cross-ratios

[Drummond et al.'07]

Duality from diagrammatical point view [Gorsky and Zhiboedov'09]

Wilson loop

$$W[\mathcal{C}_n] = \mathrm{Tr}\,\mathcal{P}\,\exp\left[ig \oint \mathrm{d}\tau \dot{x}^\mu(\tau) A_\mu(x(\tau))\right] ,$$

where $\mathcal{C}_n$ is a light-like n-edged polygonal contour, with n vertices of the polygon as $x_1, \ldots, x_n$. The length of an edge is given by the momentum of a particle in the corresponding colour-ordered scattering amplitude:

$$p_i = x_i - x_{i+1}$$

with $i = 1, \ldots, n$, $x_1 = x_{n+1}$.

Perturbative expansion:

$$\langle W[\mathcal{C}_n]\rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)} \,,$$

where $a = \frac{g^2 N}{8\pi^2}$.
First orders:

$$w_n^{(1)} = W_n^{(1)} \,, \qquad w_n^{(2)} = W_n^{(2)} - \frac{1}{2}\left(W_n^{(1)}\right)^2 \,.$$

The building block is the gluon propagator

$$G_{\mu\nu}(x) = -g_{\mu\nu}\frac{\Gamma(1-\epsilon)}{4\pi^2}(-x^2+i0)^{-1+\epsilon}\left(\mu^2\pi\right)^{\epsilon} \,.$$

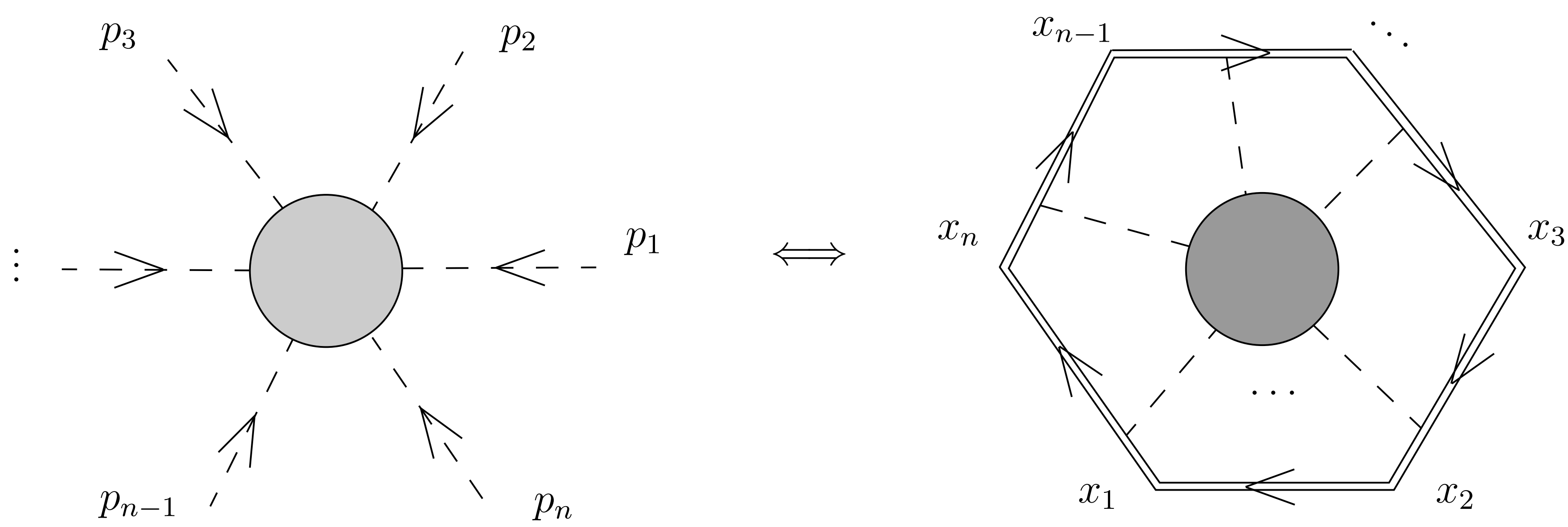
Explicitly,

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} = m_n^{(1)} - n\frac{\zeta(2)}{2} + \mathcal{O}(\epsilon)$$

with

$$m_n^{(1)} = \sum_{p,q} F^{\rm 2me}(p,q,P,Q)\,,$$

where p and q are two external momenta corresponding to two opposite massless legs, while the two remaining legs P and Q are massive.


p_3
p_2
p_1
p_{n-1}
p_n
x_{n-1}
x_n
x_3
x_1
x_2

The Wilson loop fulfils a special conformal Ward identity $\to$

[Drummond, Henn, Korchemsky and Sokatchev '07]

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

where $C_{WL}^{(2)} = C^{(2)} = -\zeta(2)^2/2$,

$$f_{WL}^{(2)}(\epsilon) = -\zeta(2) + 7\zeta(3)\epsilon - 5\zeta(4)\epsilon^2\,.$$

Duality $\to$ $R_{n,WL}^{(2)} = R_n^{(2)}$

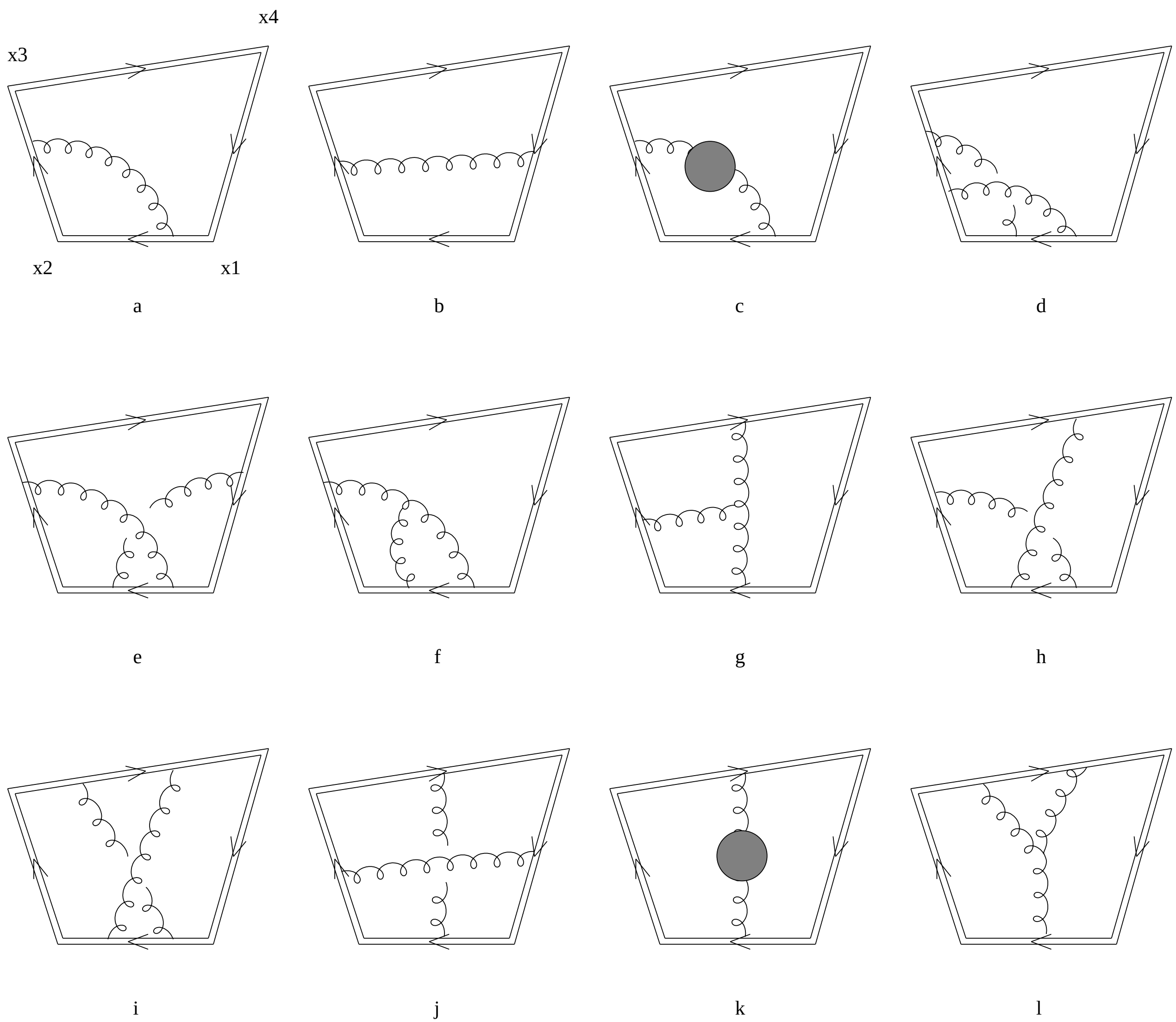
$R_{n,WL}^{(2)}$ is a function of conformally invariant cross ratios

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

For $n = 6$,

$$u_{36} = u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2}\,,\; u_{14} = u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}\,,\; u_{25} = u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}\,,$$

where $x_{ij}^2 = (x_i - x_j)^2$, $\quad x_{i,i+2}^2 = s_{i,i+1} = (p_i + p_{i+1})^2$ and $x_{i,i+3}^2 = s_{i,i+1,i+2} = (p_i + p_{i+1} + p_{i+2})^2$ (modulo 6).


x3
x4
x2
x1
a
b
c
d
e
f
g
h
i
j
k
l

Diagrams two-loop hexagon Wilson loops

[Anastasiou et al.'09]

$$
\begin{aligned}
w_6^{(2)} = \; & 2a^2 \left[\Gamma(1+\epsilon)e^{\gamma\epsilon}\right]^2 [f_H(p_1,p_2,p_3;0,p_4+p_5+p_6,0) + f_H(p_1,p_2,p_4;p_3,p_5+p_6,0) \\
& +f_H(p_1,p_2,p_5;p_3+p_4,p_6,0) + (1/3)f_H(p_1,p_3,p_5;p_4,p_6,p_2) \\
& +f_C(p_1,p_2,p_3;0,p_4+p_5+p_6,0) + f_C(p_1,p_2,p_4;p_3,p_5+p_6,0) \\
& +f_C(p_1,p_2,p_5;p_3+p_4,p_6,0) + f_C(p_1,p_2,p_6;p_3+p_4+p_5,0,0) \\
& +f_C(p_1,p_3,p_4;0,p_5+p_6,p_2) + f_C(p_1,p_3,p_5;p_4,p_6,p_2) \\
& +f_C(p_1,p_3,p_6;p_4+p_5,0,p_2) + f_C(p_1,p_4,p_5;0,p_6,p_2+p_3) \\
& +f_C(p_1,p_4,p_6;p_5,0,p_2+p_3) + f_C(p_1,p_5,p_6;0,0,p_2+p_3+p_4) \\
& +f_X(p_1,p_2;p_3+p_4+p_5+p_6,0) + f_Y(p_1,p_2;p_3+p_4+p_5+p_6,0) \\
& +f_Y(p_2,p_1;0,p_3+p_4+p_5+p_6) + f_X(p_1,p_3;p_4+p_5+p_6,p_2) \\
& +f_Y(p_1,p_3;p_4+p_5+p_6,p_2) + f_Y(p_3,p_1;p_2,p_4+p_5+p_6) \\
& +(1/2)f_X(p_1,p_4;p_5+p_6,p_2+p_3) + f_Y(p_1,p_4;p_5+p_6,p_2+p_3) \\
& +(-1/2)f_P(p_1,p_3;p_4+p_5+p_6,p_2)\, f_P(p_2,p_4;p_1+p_5+p_6,p_3) \\
& +(-1/2)f_P(p_1,p_3;p_4+p_5+p_6,p_2)\, f_P(p_2,p_5;p_1+p_6,p_3+p_4) \\
& +(-1/4)f_P(p_1,p_4;p_5+p_6,p_2+p_3)\, f_P(p_2,p_5;p_1+p_6,p_3+p_4) \\
& + \text{cyclic permutations of } (p_1,p_2,p_3,p_4,p_5,p_6)]
\end{aligned}
$$

f_H stands for a hard diagram, f_C for a curtain diagram, f_X for a cross diagram, f_Y for a Y diagram plus a self-energy diagram, f_P for a factorized cross diagram.

The basic kinematic invariants

$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{342}, s_{345}$

Other invariants are

$$
\begin{aligned}
s_{13} &= -s_{12} + s_{123} - s_{23}, \quad s_{14} = -s_{123} + s_{23} - s_{234} + s_{56}, \\
s_{15} &= -s_{16} + s_{234} - s_{56}, \quad s_{24} = -s_{23} + s_{234} - s_{34}, \\
s_{25} &= s_{16} - s_{234} + s_{34} - s_{345}, \quad s_{26} = -s_{12} - s_{16} + s_{345}, \\
s_{35} &= -s_{34} + s_{345} - s_{45}, \quad s_{36} = s_{12} - s_{123} - s_{345} + s_{45}, \\
s_{46} &= s_{123} - s_{45} - s_{56} \,.
\end{aligned}
$$

For $f_H, \dots, f_P$ parametric representations are applied

[Anastasiou et al.'09]

For example, this is part of $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$

$$F = \frac{\Gamma(2-2\epsilon)(s_{12}-s_{123}+s_{23})(s_{16}-s_{234}+s_{56})}{2\Gamma(1-\epsilon)^2}$$

$$\times \int \dots \int \left(\prod_{i=1}^{3} \mathrm{d}\alpha_i \right) \left(\prod_{i=1}^{3} \mathrm{d}\tau_i \right) \delta \left(\sum_{i=1}^{3} \alpha_i - 1 \right) (1-\tau_1) \alpha_1^{1-\epsilon} \alpha_2^{1-\epsilon} \alpha_3^{-\epsilon}$$

$$\times [-s_{12}\alpha_1\alpha_2(1-\tau_1)(1-\tau_2) - s_{123}\alpha_1\alpha_2(1-\tau_1)\tau_2 - s_{23}\alpha_1\alpha_2\tau_1\tau_2 - s_{16}\alpha_1\alpha_3\tau_1\tau_3$$

$$-s_{234}\alpha_1\alpha_3\tau_1(1-\tau_3) - s_{56}\alpha_1\alpha_3(1-\tau_1)(1-\tau_3) - s_{34}\alpha_2\alpha_3(1-\tau_2)(1-\tau_3)$$

$$-s_{345}\alpha_2\alpha_3(1-\tau_2)\tau_3 - s_{45}\alpha_2\alpha_3\tau_2\tau_3]^{2\epsilon-2}$$

The method of MB representation

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \, \Gamma(-z) \, \Gamma(\lambda + z) \, \frac{A^z}{B^{\lambda+z}} \, .$$

V.S. *Evaluating Feynman integrals* (STMP 211, Springer 2004)
Feynman Integrals Calculus (Springer 2006)

$$F = \frac{(s_{12} - s_{123} + s_{23})(s_{16} - s_{234} + s_{56})}{2\Gamma(1-\epsilon)^2\Gamma(\epsilon+1)} \frac{1}{(2\pi i)^8} \int \ldots \int \left(\prod_{i=1}^{8} \mathrm{d}z_i \Gamma(-z_i) \right) (-s_{12})^{z_1} (-s_{123})^{z_7}$$
$$\times (-s_{16})^{z_6} (-s_{23})^{z_2} (-s_{234})^{z_8} (-s_{34})^{z_3} (-s_{45})^{z_4} (-s_{56})^{z_5} (-s_{345})^{2\epsilon - z_1 - z_2 - z_3 - z_4 - z_5 - z_6 - z_7 - z_8 - 2}$$
$$\times \Gamma(\epsilon - z_1 - z_2 - z_7 - 1)\Gamma(2\epsilon - z_2 - z_4 - z_5 - z_6 - z_7 - z_8 - 1)$$
$$\times \Gamma(2\epsilon - z_1 - z_2 - z_3 - z_5 - z_7 - z_8 - 1)\Gamma(-\epsilon + z_1 + z_2 + z_5 + z_6 + z_7 + z_8 + 2)$$
$$\times \Gamma(\epsilon - z_5 - z_6 - z_8)\Gamma(-2\epsilon + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + 2)$$
$$\times \frac{\Gamma(z_1 + z_5 + z_7 + 2)\Gamma(z_2 + z_4 + z_7 + 1)\Gamma(z_2 + z_6 + z_8 + 1)\Gamma(z_3 + z_5 + z_8 + 1)}{\Gamma(2\epsilon - z_1 - z_2 - z_7)\Gamma(2\epsilon - z_5 - z_6 - z_8)\Gamma(z_1 + z_2 + z_5 + z_6 + z_7 + z_8 + 3)}$$

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \left[w_6^{(2)}(\epsilon) - f^{(2)}_{WL}(\epsilon)\, w_6^{(1)}(2\epsilon)\right]\Big|_{\epsilon=0} - C^{(2)}_{WL}$$

with $u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}$, $u_2 = \frac{s_{23}s_{56}}{s_{123}s_{234}}$, $u_3 = \frac{s_{61}s_{34}}{s_{234}s_{345}}$
Take a limit in which (i) the conformal ratios on which the remainder depends take non-trivial values and

(ii) the two-loop hexagon Wilson loop $w_6^{(2)}(\epsilon)$ is as simple as possible.
QMRK of a pair along the ladder:

$$-s_{12} \gg -s_{34}, -s_{56}, -s_{345}, -s_{123} \gg -s_{23}, -s_{45}, -s_{61}, -s_{234}$$

or

$$\{s_{34}, s_{56}, s_{123}, s_{345}\} = \mathcal{O}(\lambda)\,,$$
$$\{s_{23}, s_{45}, s_{61}, s_{234}\} = \mathcal{O}(\lambda^2)$$

with $\lambda \to 0$

The basic idea:
Take, one after another, not only this limit but also five more limits which are obtained from the first one by cyclic permutations of the external momenta $p_1, \ldots, p_6$. For example, the second limit in this series is

$$\{s_{45}, s_{61}, s_{234}, s_{123}\} \quad = \mathcal{O}(\lambda)\,,$$
$$\{s_{34}, s_{56}, s_{12}, s_{345}\} \quad = \mathcal{O}(\lambda^2)\,.$$

Evaluate the leading power asymptotics (including all the logarithms) automatically using the code `MBasymptotics` [M. Czakon]. Apply the code `barnesroutines` [D. Kosower] whenever possible to perform integrations that can be done by corollaries of Barnes lemmas.

At most threefold MB integrals are obtained after this.
The resulting Mellin-Barnes integrals are then evaluated by directly closing contours and summing up residues or by exchanging a Mellin-Barnes integration with an integral of Euler type.
Linear combination of Goncharov's multiple polylogarithms whose arguments are functions of conformal cross ratios.

$$G(\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a}\, G(\vec{w}';t) \;\;\text{and}\;\; G(\vec{0}_n;z) = \frac{1}{n!}\ln^n z\,,$$

where $\vec{w} = (a, \vec{w}')$

Our result

$$R^{(2)}_{WL,6} = \frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_3}, 1\right)$$

$$+\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_2}, \frac{1}{u_2+u_3}, 1\right)$$

$$+\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_3}, \frac{1}{u_1+u_3}, 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_3}, \frac{1}{u_2+u_3}, 1\right) + \frac{3}{2} G\left(0, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}, 1\right.$$

$$+\frac{3}{2} G\left(0, 0, \frac{1}{u_1}, \frac{1}{u_1+u_3}, 1\right) + \frac{3}{2} G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}, 1\right) + \frac{3}{2} G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}, 1\right) + \frac{3}{2} G\left(0, 0, \frac{1}{u_3}, \frac{1}{u_1+u_3}, 1\right)$$

$$+\frac{3}{2} G\left(0, 0, \frac{1}{u_3}, \frac{1}{u_2+u_3}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_2}, 1\right) + G\left(0, \frac{1}{u(1)}, 0, \frac{1}{u_1+u_2}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_3}, 1\right)$$

$$+G\left(0, \frac{1}{u(1)}, 0, \frac{1}{u_1+u_3}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1+u_2}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1+u_3}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_1+u_2}, \right.$$

$$-\frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_3}, \frac{1}{u_1+u_3}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1}, 1\right) + G\left(0, \frac{1}{u(2)}, 0, \frac{1}{u_1+u_2}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_3}, 1\right)$$

$$+G\left(0, \frac{1}{u(2)}, 0, \frac{1}{u_2+u_3}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_2}, \frac{1}{u_1}, \frac{1}{u_1+u_2}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_2}, \frac{1}{u_2}, \frac{1}{u_1+u_2}, 1\right) - \frac{1}{2} G\left(0, \frac{1}{u_2}, \frac{1}{u_2}, \frac{1}{u_2+u_3}, \right.$$

$$-\frac{1}{2} G\left(0, \frac{1}{u_2}, \frac{1}{u_3}, \frac{1}{u_2+u_3}, 1\right) + \frac{1}{4} G\left(0, \frac{u_2-1}{u_1+u_2-1}, 0, \frac{1}{1-u_1}, 1\right) + \frac{1}{4} G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 0, 1\right)$$

$$+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1},1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1\right)$$

$$-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1,1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1},1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2},1\right)+G\left(0,\frac{1}{u(3)},0,\frac{1}{u_1+u_3},1\right)$$

$$+G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3},1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3},1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3},1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3}\right.$$

$$-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3},1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3},1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0,1\right)$$

$$-\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1,1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3},1\right)+\ldots$$

$\sim$ 1000 terms.

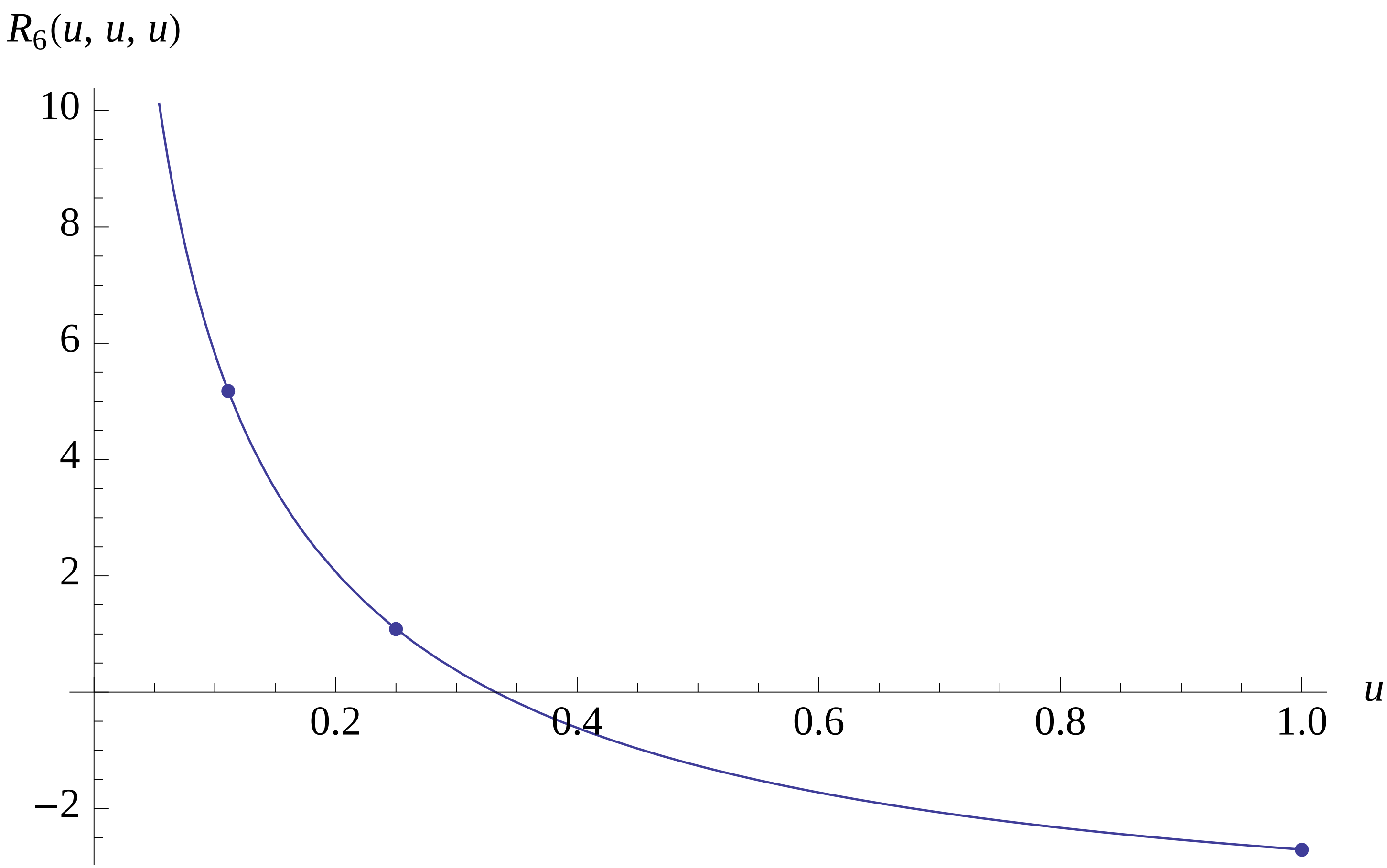
Numerical checks by `FIESTA` [A. Smirnov and Tentyukov'09]
An agreement with other numerical results [Anastasiou et al.'09]

Let us choose $u_1 = u_2 = u_3 = u$

$$\begin{aligned} R^{(2)}_{WL,6}(1,1,1) &= -\tfrac{\pi^4}{36} \simeq -2.70581\ldots\,, \\ \lim_{u\to\infty} R^{(2)}_{WL,6}(u,u,u) &= -\tfrac{\pi^4}{144} \simeq -0.67645\ldots\,, \end{aligned}$$

in agreement with Anastasiou et al.

$$\lim_{u\to 0} R^{(2)}_{WL,6}(u,u,u) = \frac{\pi^2}{8}\ln^2 u + \frac{17\pi^4}{1440} + \mathcal{O}(u)\,.$$


$R_6(u,u,u)$
10
8
6
4
2
−2
0.2
0.4
0.6
0.8
1.0
u

Open problems and perspectives

A small disagreement at small $u_1 = u_2 = u_3 = u$ with an AGM formula

[Alday, Gaiotto and Maldacena,'09]

Amplitudes $\leftrightarrow$ Wilson loops?

$n = 8$ and a comparison with an AM suggestion

[Alday and Maldacena,'09]

Starting from $n = 6$ the structure of the planar MHV amplitudes is more complicated.
Exponentiation?

Not only virtual corrections but also real radiation in $N = 4$ SYM $\rightarrow$ D. Kazakov's talk

backup slides

The leading-N_c contributions to the L-loop $SU(N_c)$ gauge-theory n-point amplitudes:

$$\mathcal{M}_n^{(L)} = g^{n-2}\left[\frac{2e^{-\epsilon\gamma}g^2N_c}{(4\pi)^{2-\epsilon}}\right]^L \sum_\rho \mathrm{Tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(n)}}) M_n^{(L)}(\rho(1),\rho(2),\ldots,\rho($$

where γ is Euler's constant, and the sum runs over non-cyclic permutations of the external legs.