Gauge Fields. Yesterday, Today, Tomorrow

dedicated to the 70-th anniversary of

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Moscow, January 19 – 24, 2010.

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Higher covariant derivative regularization for calculations in supersymmetric theories

Regularization for the supersymmetric theories

Quantum corrections in supersymmetric theories are investigated for a long time, for example in the papers

L.V.Avdeev, O.V.Tarasov, Phys.Lett. 112B, (1982),356;
A.Parkes, P.West, Phys.Lett. 138B, (1983), 99;
I.Jack, D.R.T.Jones, C.G.North, Phys.Lett B386, (1996), 138; Nucl.Phys. B486 (1997), 479;
I.Jack, D.R.T.Jones, A.Pickering, Phys.Lett. B435, (1998), 61.

Most calculations were made with the dimensional reduction

W.Siegel, Phys.Lett. 84B, (1979), 193; 94B, (1980), 37.

(The dimensional regularization breaks the supersymmetry.) With the dimensional reduction the β-function was calculated even in the four-loop approximation. However, the dimensional reduction is inconsistent from the mathematical poit of view and can lead to some problems in higher loops.

The higher covariant derivatives

A consistent regularization, which does not break the supersummetry is the higher covariant derivative regularization, proposed by A.A.Slavnov:

A.A.Slavnov, Nucl.Phys., **B31**, (1971), 301; Theor.Math.Phys. **13**, (1972), 1064.

It was generalized to the supersymmetric case in the papers

V.K.Krivoshchekov, Theor.Math.Phys. **36**, (1978), 745; P.West, Nucl.Phys. B268, (1986), 113.

The first (one-loop) calculation with the higher derivative regularization was made for the (nonsupersymmetric) Yang–Mills theory in

C.Martin, F.Ruiz Ruiz, Nucl.Phys. **B 436**, (1995), 645.

Taking into account correction, made in

M.Asorey, F.Falceto, Phys.Rev **D 54**, (1996), 5290; T.Bakeyev, A.Slavnov, Mod.Phys.Lett. **A11**, (1996), 1539.

the result coincided with the usual β-function of the Yang–Mills theory.

$N = 1$ supersymmetric theories

N=1 supersymmetric Yang-Mills theory with matter in the massless case is described by the action

$$S = \frac{1}{2e^2}\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W_a C^{ab} W_b + \frac{1}{4}\int d^4x\,d^4\theta\,(\phi^*)^i (e^{2V})_i{}^j \phi_j +$$
$$+\Big(\frac{1}{6}\int d^4x\,d^2\theta\,\lambda^{ijk}\phi_i\phi_j\phi_k + \mathrm{h.c.}\Big),$$

where ϕ_i are chiral scalar matter superfields, V is a real scalar gauge superfield, and the supersymmetric gauge field stress tensor is given by

$$W_a = \frac{1}{8}\bar{D}^2\Big[e^{-2V}D_a e^{2V}\Big].$$

The action is invariant under the gauge transformations

$$e^{2V} \to e^{i\Lambda^+}e^{2V}e^{-i\Lambda}; \qquad \phi \to e^{i\Lambda}\phi$$

if $(T^A)_m{}^i\lambda^{mjk} + (T^A)_m{}^j\lambda^{imk} + (T^A)_m{}^k\lambda^{ijm} = 0.$

Background field method

We use the background field method: $e^{2V} \to e^{2V'} \equiv e^{\mathbf{\Omega}^+} e^{2V} e^{\mathbf{\Omega}}$, where $\mathbf{\Omega}$ is a background field. Background covariant derivatives are given by

$$\boldsymbol{D} \equiv e^{-\mathbf{\Omega}^+}\frac{1}{2}(1+\gamma_5)De^{\mathbf{\Omega}^+}; \qquad \bar{\boldsymbol{D}} \equiv e^{\mathbf{\Omega}}\frac{1}{2}(1-\gamma_5)De^{-\mathbf{\Omega}};$$

$$\boldsymbol{D}_\mu \equiv -\frac{i}{4}(C\gamma^\mu)^{ab}\Big\{\boldsymbol{D}_a, \bar{\boldsymbol{D}}_b\Big\}.$$

The background gauge invariance

$$\phi \to e^{i\Lambda}\phi; \quad V \to e^{iK} V e^{-iK}; \quad e^{\mathbf{\Omega}} \to e^{iK} e^{\mathbf{\Omega}} e^{-i\Lambda}; \quad e^{\mathbf{\Omega}^+} \to e^{i\Lambda^+} e^{\mathbf{\Omega}^+} e^{-iK},$$

where K is an arbitrary real superfield, and Λ is a background-chiral superfield.

This invariance allows to choose $\mathbf{\Omega} = \mathbf{\Omega}^+ = \mathbf{V}$.

It is desirable to fix a gauge and to introduce a regularization in such a way, that the background gauge invariance will be unbroken.

Quantization

The gauge is fixed by adding the following term:

$$S_{gf} = -\frac{1}{32e^2}\,\mathrm{tr}\int d^4x\,d^4\theta\,\Big(V\boldsymbol{D}^2\bar{\boldsymbol{D}}^2V + V\bar{\boldsymbol{D}}^2\boldsymbol{D}^2V\Big).$$

(Then the terms, quadratic in the quantum field, have the simplest form.)

The corresponding ghost Lagrangian is

$$S_c = i\,\mathrm{tr}\int d^4x\,d^4\theta\,\Big\{(\bar{c}+\bar{c}^+)V\Big[(c+c^+) + \mathrm{cth}\,V(c-c^+)\Big]\Big\}.$$

Also it is necessary to add the Nielsen-Kallosh ghosts

$$S_B = \frac{1}{4e^2}\mathrm{tr}\int d^4x\,d^4\theta\,B^+e^{\boldsymbol{\Omega}^+}e^{\boldsymbol{\Omega}}\,B.$$

Higher derivative regularization

To regularize the theory we use the higher covariant derivative regularization.

For a theory with the nontrivial cubic superpotential it is also necessary to introduce the higher covariant derivative term for the matter superfields. We add to the action the term

$$S_\Lambda = \frac{1}{2e^2}\,\mathrm{tr}\,\mathrm{Re}\int d^4x\,d^4\theta\,V\,\frac{(\boldsymbol{D}_\mu^2)^{n+1}}{\Lambda^{2n}}V + \frac{1}{8}\int d^4x\,d^4\theta\,\Big((\phi^*)^i \times$$

$$\times\Big[e^{\boldsymbol{\Omega}^+}\frac{(\boldsymbol{D}_\alpha^2)^m}{\Lambda^{2m}}e^{\boldsymbol{\Omega}}\Big]_i{}^j\phi_j + (\phi^*)^i\Big[e^{\boldsymbol{\Omega}^+}\frac{(\boldsymbol{D}_\alpha^2)^m}{\Lambda^{2m}}e^{\boldsymbol{\Omega}}\Big]_i{}^j\phi_j\Big).$$

Presence of the higher derivatives in the matter kinetic terms makes the calculations much more complicated.

After adding of the term with the higher derivatives divergences remain only in the one-loop approximation.

Higher derivative regularization

In order to regularize the remaining one-loop divergences, it is necessary to introduce Pauli-Villars determinants into the generating functional

L.D.Faddeev, A.A.Slavnov, Gauge fields, introduction to quantum theory, Benjamin, Reading, 1990.

$$Z[J,\mathbf{\Omega}] = \int D\mu \prod_I \Big(\det PV(V,\mathbf{V},M_I)\Big)^{c_I} \times$$

$$\times \exp\Big\{iS + iS_\Lambda + iS_{gf} + iS_B + iS_{gh} + \mbox{Sources}\Big\},$$

where the coefficients satisfy the conditions $\sum\limits_I c_I = 1; \qquad \sum\limits_I c_I M_I^2 = 0.$

It is convenient to write the Pauli–Villars determinants as

$$\det PV(V,\mathbf{V},M) = \Big(\int D\Phi^* D\Phi e^{iS_{PV}}\Big)^{-1}.$$

In order to cancel the remaining one-loop divergences of the theory with the higher derivative term the Pauli–Villars action S_{PV} should contain the higher derivatives.

Pauli–Villars fields

We considered the following form of the Pauli–Villars action:

$$S_{PV} = \frac{1}{8}\int d^4x\, d^4\theta\, \Big((\Phi^*)^i \Big[e^{\mathbf{\Omega}^+}\Big(1+\frac{(\boldsymbol{D}^2_\alpha)^m}{\Lambda^{2m}}\Big)e^{\mathbf{\Omega}}\Big]_i{}^j \Phi_j +$$

$$+(\Phi^*)^i \Big[e^{\mathbf{\Omega}^+}\Big(1+\frac{(\boldsymbol{D}^2_\alpha)^m}{\Lambda^{2m}}\Big)e^{\mathbf{\Omega}}\Big]_i{}^j \Phi_j\Big) + \Big(\frac{1}{4}\int d^4x\, d^4\theta\, M^{ij}\times$$

$$\times(e^{\mathbf{\Omega}}\Phi)_i(e^{\mathbf{\Omega}}\Phi)_j + \text{h.c.}\Big).$$

(A regularized one-loop diagram with cubic matter vertex is finite.)

In order to obtain the gauge invariance the mass should satisfy

$$(T^A)_k{}^i M^{kj} + (T^A)_k{}^j M^{ki} = 0.$$

Also we assume that

$$M^{ij}M^*_{jk} = M^2\delta^i_k \qquad M^{ij} = a^{ij}\Lambda,$$

where a^{ij} are constants. (There is the only dimensionful parameter Λ.)

Two-loop β-function for $N=1$ supersymmetric Yang-Mills theory

Two-loop calculation gives the following result:

$$\beta(\alpha) = -\frac{3\alpha^2}{2\pi}C_2 + \alpha^2 T(R) I_0 + \alpha^3 C_2^2 I_1 + \frac{\alpha^3}{r} C(R)_i{}^j C(R)_j{}^i I_2 +$$
$$+\alpha^3 T(R) C_2 I_3 + \alpha^2 C(R)_i{}^j \frac{\lambda^*_{jkl}\lambda^{ikl}}{4\pi r} I_4 + \ldots,$$

where we do not write the integral for the one-loop ghost contribution and the integrals I_0–I_4 are given below, and the following notation is used:

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA}.$$

Taking into account Pauli–Villars contributions,

$$I_i = I_i(0) - \sum_I I_i(M_I), \qquad i = 0, 2, 3$$

where I_i are given by

Factorization of integrands into total derivatives

$$I_0(M) = 8\pi \int \frac{d^4q}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{1}{q^2} \frac{d}{dq^2} \Bigg\{ \frac{1}{2} \ln\left(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2\right) + \\
+ \frac{M^2}{2(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2)} - \frac{mq^{2m}/\Lambda^{2m} q^2 (1+q^{2m}/\Lambda^{2m})}{q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2} \Bigg\};$$

$$I_1 = 96\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{1}{k^2} \frac{d}{dk^2} \Bigg\{ \frac{1}{q^2(q+k)^2(1+q^{2n}/\Lambda^{2n})} \times \\
\times \frac{1}{(1+(q+k)^{2n}/\Lambda^{2n})} \left(\frac{n+1}{(1+k^{2n}/\Lambda^{2n})} - \frac{n}{(1+k^{2n}/\Lambda^{2n})^2} \right) \Bigg\};$$

$$I_2(M) = -64\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{1}{q^2} \frac{d}{dq^2} \Bigg\{ \frac{q^2}{k^2(1+k^{2n}/\Lambda^{2n})} \times \\
\times \frac{(1+(q+k)^{2m}/\Lambda^{2m})}{((q+k)^2(1+(q+k)^{2m}/\Lambda^{2m}) + M^2)} \Bigg[\frac{q^2(1+q^{2m}/\Lambda^{2m})^3}{(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2)^2} + \\
+ \frac{mq^{2m}/\Lambda^{2m}}{q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2} - \frac{2mq^{2m}/\Lambda^{2m} M^2}{(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2)^2} \Bigg] \Bigg\};$$

Factorization of integrands into total derivatives

$$I_3(M) = 16\pi^2 \int \frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d}{d\ln\Lambda}\Bigg\{\frac{\partial}{\partial q_\alpha}\Bigg[\frac{k_\alpha(1+q^{2m}/\Lambda^{2m})}{(q^2(1+q^{2m}/\Lambda^{2m})^2+M^2)}\times$$

$$\times\frac{1}{(k+q)^2(1+(q+k)^{2n}/\Lambda^{2n})}\Bigg(-\frac{(1+k^{2m}/\Lambda^{2m})^3}{(k^2(1+k^{2m}/\Lambda^{2m})^2+M^2)^2}+$$

$$+\frac{mk^{2m}/\Lambda^{2m}}{k^2(1+k^{2m}/\Lambda^{2m})^2+M^2}-\frac{2mk^{2m}/\Lambda^{2m}M^2}{(k^2(1+k^{2m}/\Lambda^{2m})^2+M^2)^2}\Bigg)\Bigg]-$$

$$-\frac{1}{k^2}\frac{d}{dk^2}\Bigg[\frac{2(1+q^{2m}/\Lambda^{2m})(1+(q+k)^{2m}/\Lambda^{2m})}{(q^2(1+q^{2m}/\Lambda^{2m})^2+M^2)\,((q+k)^2(1+(q+k)^{2m}/\Lambda^{2m})^2+M^2)}\times$$

$$\times\Bigg(\frac{1}{(1+k^{2n}/\Lambda^{2n})}+\frac{nk^{2n}/\Lambda^{2n}}{(1+k^{2n}/\Lambda^{2n})^2}\Bigg)\Bigg]\Bigg\};$$

$$I_4 = 64\pi^2\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d}{d\ln\Lambda}\frac{1}{q^2}\frac{d}{dq^2}\Bigg[\frac{1}{k^2(q+k)^2(1+k^{2m}/\Lambda^{2m})}\times$$

$$\times\frac{1}{(1+(q+k)^{2m}/\Lambda^{2m})}\Bigg(\frac{1}{(1+q^{2m}/\Lambda^{2m})}+\frac{mq^{2m}/\Lambda^{2m}}{(1+q^{2m}/\Lambda^{2m})^2}\Bigg)\Bigg].$$

Two-loop β-function for $N=1$ supersymmetric Yang-Mills theory

The integrals can be calculated using the identity

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{d}{dk^2}f(k^2) = \frac{1}{16\pi^2}\Big(f(k^2=\infty)-f(k^2=0)\Big).$$

(This is a total derivative in the four-dimensional spherical coordinates.)

The result for the two-loop β-function is given by

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha^3}{(2\pi)^2}\Big(-3C_2^2 + T(R)C_2 + \\ +\frac{2}{r}C(R)_i{}^j C(R)_j{}^i\Big) - \frac{\alpha^2 C(R)_i{}^j \lambda^*_{jkl}\lambda^{ikl}}{8\pi^3 r} + \ldots.$$

Two-loop β-function for $N = 1$ supersymmetric Yang-Mills theory

Comparing the result with the one-loop anomalous dimension

$$\gamma_i{}^j(\alpha) = -\frac{\alpha C(R)_i{}^j}{\pi} + \frac{\lambda^*_{ikl}\lambda^{jkl}}{4\pi^2} + \ldots,$$

gives the exact NSVZ β-function in the considered approximation.

$$\beta(\alpha) = -\frac{\alpha^2\Big[3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big)\Big]}{2\pi(1 - C_2\alpha/2\pi)}.$$

V.Novikov, M.A.Shifman, A.Vainstein, V.Zakharov, Nucl.Phys. B 229, (1983), 381; Phys.Lett. 166B, (1985), 329; M.Shifman, A.Vainshtein, Nucl.Phys. B 277, (1986), 456.

(The result also agrees with the DRED calculations.)

Three-loop calculation for SQED

The notation is

$$\Gamma^{(2)} = \int \frac{d^4p}{(2\pi)^4}\, d^4\theta \Big(-\frac{1}{16\pi} \mathbf{V}(-p)\, \partial^2 \Pi_{1/2} \mathbf{V}(p)\, d^{-1}(\alpha, \mu/p) + $$
$$+ \frac{1}{4} (\phi^*)^i(-p,\theta)\, \phi_j(p,\theta)\, (ZG)_i{}^j(\alpha,\mu/p)\Big). \qquad (1)$$

The main result: (It was obtained as the equality of some well defined integrals)

$$\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0} = -\frac{d}{d\ln\Lambda}\alpha_0^{-1}(\alpha,\mu/\Lambda) =$$
$$= \frac{1}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big) = \frac{1}{\pi} + \frac{1}{\pi}\frac{d}{d\ln\Lambda}\Big(\ln ZG(\alpha,\mu/q) -$$
$$- \ln Z(\alpha,\Lambda/\mu)\Big)\Big|_{q=0} = \frac{1}{\pi}\Big(1 - \gamma\Big(\alpha_0(\alpha,\Lambda/\mu)\Big)\Big).$$

The reason is that the integrands are again total derivatives.

Toward the explanation

For simplicity we will consider Abelian case (SUSY QED).

The two-point Green function of the gauge superfield can be written as

$$\frac{\delta\Gamma}{\delta\mathbf{V}_y\delta\mathbf{V}_x} = \frac{e}{2}\frac{\delta}{\delta\mathbf{V}_y}\Big\langle \phi^*_x e^{2V'_x}\phi_x + \tilde\phi^*_x e^{-2V'_x}\tilde\phi_x\Big\rangle.$$

Introducing auxiliary sources

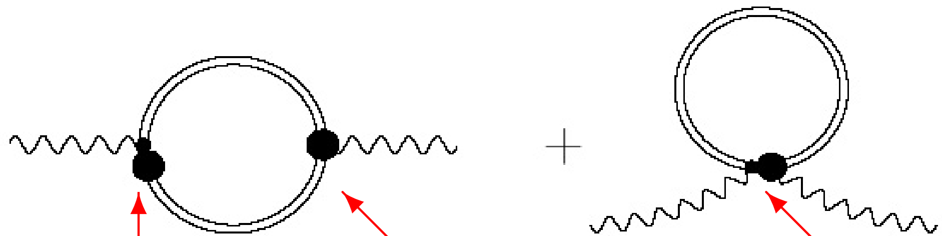
$$S_{\text{Source}} = \frac{1}{4}\int d^8x\,(\phi_0^* e^{2V'}\phi + \tilde\phi_0^* e^{-2V'}\tilde\phi + \text{h.c.})$$

so that

$$-\frac{D^2}{2}\frac{\delta\Gamma}{\delta\phi_0^*} = -\frac{D^2}{8}\Big\langle e^{2V}\phi\Big\rangle = \frac{\delta\Gamma}{\delta\phi^*},$$

it is possible to write the Schwinger-Dyson equation in the following graphical form:

$$\frac{\delta^2\Gamma}{\delta \mathbf{V}_x \delta \mathbf{V}_y} =$$



Here (in the massless case for simplicity) the vertexes are given by

$$\frac{\delta^2\Gamma}{\delta\phi_{0x}^*\delta\phi_y} = -\frac{1}{8}G(\partial^2)\bar{D}_x^2\delta_{xy}^8; \qquad \frac{\delta^3\Gamma}{\delta\mathbf{V}_x\delta\phi_z^*\delta\phi_w}; \qquad \frac{\delta^3\Gamma}{\delta\mathbf{V}_x\delta\phi_{0z}^*\delta\phi_w},$$

and the propagator is

$$\frac{\delta^2\Gamma}{\delta\phi_x^*\delta\phi_y} = -\frac{D_x^2\bar{D}_x^2}{4\partial^2 G(\partial^2)}\delta_{xy}^8.$$

This result is exact. Each effective diagram is an infinite series of ordinary diagrams. It is possible to check that they are equivalent to ordinary Feynman diagrams in each order of the perturbation theory.

Calculating matter contribution by Schwinger-Dyson equations and Slavnov-Taylor identities

Expressions for vertexes can be found by solving Slavnov-Taylor identities:

$$\frac{\delta^3\Gamma}{\delta\mathbf{V}_y\delta\phi^*_{0z}\delta\phi_x}\Big|_{p=0} = e\Big[-2F\partial^2\Pi_{1/2y}\Big(\bar{D}^2_y\delta^8_{xy}\delta^8_{yz}\Big) + \frac{1}{8}fD^bC_{bc}\bar{D}^2_y$$

$$\times\Big(\bar{D}^2_y\delta^8_{xy}D^c_y\delta^8_{yz}\Big) + \frac{i}{16}\partial^\mu_x G'\bar{D}\gamma^\mu\gamma_5 D_y\Big(\bar{D}^2_y\delta^8_{xy}\delta^8_{yz}\Big) - \frac{1}{4}G\bar{D}^2_y\delta^8_{xy}\delta^8_{yz}\Big];$$

$$\frac{\delta^3\Gamma}{\delta\mathbf{V}_y\delta\phi^*_{z}\delta\phi_x}\Big|_{p=0} = e\Big[F\partial^2\Pi_{1/2y}\Big(\bar{D}^2_y\delta^8_{xy}D^2_y\delta^8_{yz}\Big) -$$

$$-\frac{i}{32}\partial^\mu_x G'\bar{D}\gamma^\mu\gamma_5 D_y\Big(\bar{D}^2_y\delta^8_{xy}D^2_y\delta^8_{yz}\Big) + \frac{1}{8}G\bar{D}^2_y\delta^8_{xy}D^2_y\delta^8_{yz}\Big].$$

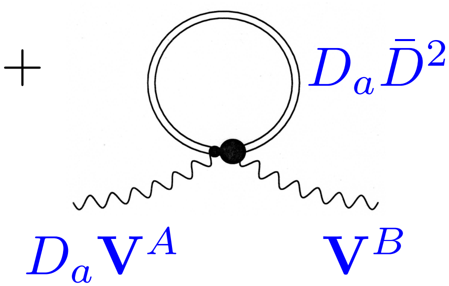
where all functions depend on ∂^2_x.

Both vertexes are defined by the same diagrams, but in the first case one of the external lines is not chiral.

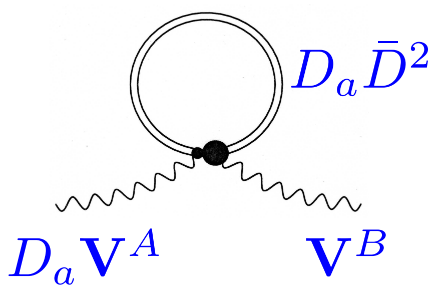
Substituting solution of these identities we obtain (in the massless case for simplicity)

$$\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\lambda_0,\Lambda/p)-\alpha_0^{-1}\Big)\Big|_{p=0} = 8\pi\frac{d}{d\ln\Lambda}\int\frac{d^4q}{(2\pi)^4}\times$$

$$\times\frac{1}{q^2}\frac{d}{dq^2}\Big(\ln(q^2G^2)\Big)+$$



In the three-loop approximation



$$=0$$

(This diagram is also given by the integral of a total derivative.)

Possibly this can be proven rewriting this effective diagram as a sum of two-loop effective diagrams. However, it is possible that in higher loops this diagram is not zero or is not given by the integral of total derivative.

Conclusion and open questions

- ✓ With the higher derivative regularization integrals, defining the β-function, are the integrals of total derivatives. This allows to calculate one of the loop integrals analytically. This seems to produces the exact NSVZ β-function.
- ✓ In order to explain factorization of integrands into total derivatives one can try to substitute solutions of the Slavnov–Taylor identities into the Schwinger–Dyson equations. However, there is a contribution of the transversal part of the effective vertex, which can not be factorized into the integral of total derivative by this method.
- ✓ Possibly, rewriting this contribution as a sum of effective diagrams with more loops can lead to the explanation.

Thank you for the attention!